To Prove Four Color Theorem

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1 Abstract

Four color theorem states that a planar graph is 4-colorable which still does not have one mathematical proof since 1852. In this paper, we will prove that a planar graph G has a color assignment using ≤ 4 colors in which G's perimeter is assigned ≤ 3 colors. I.e., every planar graph is 4-colorable.

2 Introduction

It is known that four color theorem is one special case of Hadwiger conjecture [1] when $k = 5$. I.e., if a graph has its chromatic number 5, then there is one K_5 minor in it. And the case when $k = 4$ has been proved, i.e. a chromatic number 4 graph has one K_4 minor [5].

The four color theorem has been proved assisted by computer for the first time in 1976 by Kenneth Appel and Wolfgang Haken. A simpler proof using the same idea and also relied on computer was given in 1997 by Robertson, Sanders, Seymour, and Thomas. Additionally in 2005, the theorem was proven by Georges Gonthier with general purpose theorem proving software which is also relied on computer. All these proofs have one thing in common that they are all complicated computer-assisted proofs which render it unreadable and uncheckable by hand. *None of such proofs is a mathematical proof.*

In this paper, we will prove that a planar graph G has a color assignment using ≤ 4 colors in which G's perimeter is assigned ≤ 3 colors. Hence we prove that every planar graph is 4-colorable. Moreover, we claim that by using results of [**?**,**?**], this proof can be generalized to prove Hadwiger Conjecture.

In Section 3, necessary terminologies and definitions are introduced. In Section 4, some results are proved prepared for later use in proof of four color theorem.

3 Terminology Definition and Preliminary Results

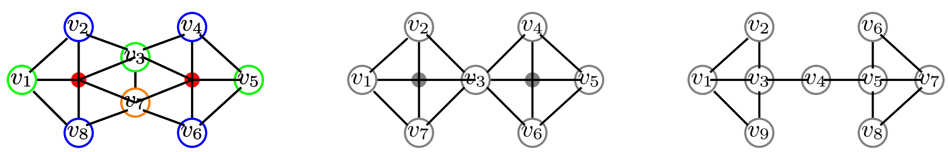
In this section, Conventional graph theory terminology applied. In Subsection 3.1, *Perimeter Trace* of a planar graph and *cluster* are defined and some their properties are introduced. In Subsection 3.2, *color collections* is defined and analyzed.

Definition 1. *To graph $G(V, E)$, one color assignment can be treated as a group of partitions of V, in which one partition is an independent set, and every partition is assigned with one different color.*

In this paper, we often use cl to denote one color assignment and also use cl to represent colors used in cl. And for convenience, we use integers to represent colors. Then we can say there is one color assignment or a set of colors $cl = \{1, 2, ..., l\}, |cl| = l$. In cl, a color used on vertex v is represented by $color_{cl}(v)$, when there is confusion, also use $color(v)$ directly.

The terminologies below are used in this paper. Given a graph $G = (V, E)$, a subgraph $G_s = (V_s, E_s)$ of G, vertex $v \in V$, and a set of vertices W: $G'_s(V'_s, E'_s) = G_s \cup v$ means that in $G'_s(V'_s, E'_s)$, $V'_s = V_s \cup v$ and $E'_s = E_s \cup \{edges\ from\ v\ to\ V_s\ in\ G\}$. $G'_s(V'_s, E'_s) = G_s \cup W$ means that in $G'_s(V'_s, E'_s)$, $V'_s = V_s \cup W$ and $E'_s = E_s \cup \{edges\ from\ W\ to\ W \cup V_s\ in\ G\}$.

3.1 Perimeter Trace



(a) (b) (c)

Fig. 1. Examples of Perimeter Trace

Definition 2. *In a connected graph $G(V, E)$, define its perimeter trace as a walking on a series of vertices $\{v_1, v_2, ..., v_x, v_1\}$, in which if there are $v_i, v_j, v_k, v_l, i < j < k < l$ then every path between v_i, v_k intersects with every path between v_j, v_l by assuming there is edge for every pair of $v_y, v_{(y+1) mod\ (x+1)}$.*

The beginning and ending of a perimeter trace are considered the same vertex. For convenience, we say two vertices appear continuously if in a perimeter trace one follow another one without separation by other vertices. In a planar graph, if we trace its perimeter, we can get a perimeter trace. For example, in Figure 1.a, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_1\}$ is a perimeter trace. When the planar graph is 1-connected, as shown in Figure 1.b, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_3, v_7, v_1\}$ is a perimeter trace where v_3 appears twice. In Figure 1.c, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_5, v_4, v_3, v_9, v_1\}$ is a perimeter trace. It is worthy to notice that in a perimeter trace, one vertex may appear more than one time.

Observation 1 *In a perimeter trace s of graph $G(V, E)$, if one vertex appears more than 1 time noncontinuously, the vertex is a cut vertex.*

A roundly continuous part of a perimeter trace S is called a *cluster* of S. *One empty set of vertices can be a cluster of any perimeter trace.* If we decompose a perimeter trace S into a set of clusters $\Upsilon = \{\Upsilon_1, \Upsilon_2, ..., \Upsilon_x\}$ by order *on the perimeter trace* where $\bigcup_{i=1}^{x} \Upsilon_i = s$. cs is called clusters of s if the ending of Υ_i may only overlap on ≤ 1 vertex with the beginning of $\Upsilon_{i+1 \ mod \ x}$.

For example, in Figure 1.a, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_1\}$ is a perimeter trace. Because one cluster can be chose roundly, "v_7, v_8, v_1" is one cluster. $\{$"v_1, v_2, v_3", "v_4, v_5", "v_6, v_7, v_8"$\}$ and $\{$"v_1, v_2, v_3", "v_3", "v_3", "v_3", "v_4, v_5", "v_6, v_7, v_8"$\}$ are its two sets of clusters.

Proposition 1. *If connected graph $G(V, E)$ is planar, then a walking generates a perimeter trace if and only if the walking is on a perimeter. If S is G's perimeter trace and $u \in S$ which is not a cut vertex, after deleting u from G, $\{S \setminus u\} \cup \{N(u) \setminus S\}$ is a perimeter trace.*

Proof. This follows from known properties of planar graph. After deleting $u \in S$ from G, the new perimeter trace is the same except u is replace by $N(u) \setminus S$. $\square$

3.2 Color Collections

Given $m \geq 0$ colors, $n \leq m$, there are C_m^n combinations of n colors. Assume we are using colors $M = \{1, 2, ..., m\}$.

Definition 3. *If there is one set of colors $L \subseteq M$ with $|L| = l \leq n$, we say L can be extended to be n colors by adding colors from $M \setminus L$, and we call those collections as n-collection of L respect to M.*

Easy to see, there are C_{m-l}^{n-l} different n-collections of L respect to M. If $l = 0$, then such collections are called M's n-collections. We say a cluster is colored by a n-collection, if the colors used on the cluster is a subset of a n-collection of M. Here we use $cn(\Upsilon_i)$ to represent the n-collection coloring of cluster Υ_i.

In this paper below, we will assume $M = \{1, 2, 3, 4\}$ where $m = 4$, and $n = 3$ color collections are used. Without confusion, when we say color collection, it means 3-collection respect to M. Hence there are $C_4^3 = 4$ 3-collections such that $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$.

If $l = 2$ with color-set $c = \{1, 2\}$, then c can be extended in $C_{4-2}^{3-2} = 2$ ways to be $\{1, 2, 3\}, \{1, 2, 4\}$. *Below if we are talking about $n - collection$, a color set whose cardinality is $\leq n$ will be considered as equivalent with all its collections.* I.e., if we are talking about $3 - collection$, color set $\{1, 2\}$ is equivalent with $\{1, 2, 3\}, \{1, 2, 4\}$. So if we are talking about the cardinality of a set of $n - collection$, all color-sets are extended to be $n - collections$. I.e., when talking $3 - collection$, $\{1, 2\}$ has its cardinality as 2.

For example, in Figure 1.a we use colors $M = \{1 = red, 2 = green, 3 = blue, 4 = orange\}$ and 3-collections to color clusters $\{v_8, v_1, v_2\}$, $\{v_4, v_5, v_6\}$, $\{v_7\}$ on perimeter trace $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_1\}$. $cn(\{v_8, v_1, v_2\}) = \{2, 3\} = \{\{2, 3, 1\}, \{2, 3, 4\}$, means cluster $\{v_8, v_1, v_2\}$ can be considered being colored

by 3-collection $\{2,3,1\}$ or $\{2,3,4\}\}$. Also there are $cn(\{v_4,v_5,v_6\}) = \{2,3\} = \{\{2,3,1\},\{2,3,4\}\}$, and $cn(\{v_7\}) = \{4\} = \{\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$.

Definition 4. *Given two sets of collections C_1, C_2, if $C_1 \setminus C_2 \neq \emptyset$ and $C_2 \setminus C_1 \neq \emptyset$, then C_1, C_2 are inconsistent; otherwise, they are consistent.*

4 Valid Constraints on Perimeter Trace

By Definition 3, we can compare two clusters by their color collections. *When we are comparing two clusters, every cluster is considered to be assigned with one n-collection.* We call a color constraint on two clusters Υ_i, Υ_j as $cn(\Upsilon_i) = cn(\Upsilon_j)$ or $cn(\Upsilon_i) \neq cn(\Upsilon_j)$. For example, in Figure 1.a as explained in the end of Section 3.2, setting $\Upsilon_i = \{v_8, v_1, v_2\}$ and $\Upsilon_j = \{v_4, v_5, v_6\}$, if $cn(\Upsilon_i) = cn(\Upsilon_j)$, there is is $cn(\Upsilon_i) = cn(\Upsilon_j) = \{1,2,3\}$ or $cn(\Upsilon_i) = cn(\Upsilon_j) = \{2,3,4\}$; if $cn(\Upsilon_i) \neq cn(\Upsilon_j)$, there is $cn(\Upsilon_i) = \{1,2,3\}, cn(\Upsilon_j) = \{2,3,4\}$ or $cn(\Upsilon_i) = \{2,3,4\}, cn(\Upsilon_j) = \{1,2,3\}$.

4.1 Collection Constraints

In Figure 1, constraints between clusters are demonstrated by graphs. In Figure 1.a, Figure 1.b and Figure 1.c clusters Υ_i, Υ_j are indicated by dashed lines; and in Figure 1.c, $\Upsilon_j = \{v\}$ which which is indicated by dashed circle; in Figure 1.d indicated by dashed line and $\Upsilon_j = \{v_4\}$.

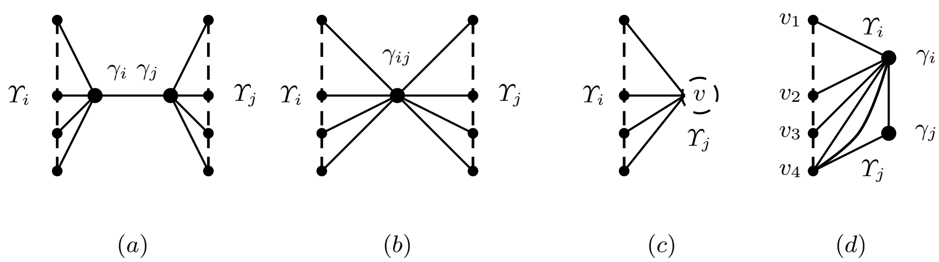


Fig. 2. Examples of Color Collections

Observation 2 *If Figure 1.a is colored with ≤ 4 colors and , then constraint $cn(\Upsilon_i) \neq cs(\Upsilon_j)$ is satisfied; if Figure 1.b is colored with ≤ 4 colors, then constraint $cn(\Upsilon_i) = cs(\Upsilon_j)$ is satisfied; if Figure 1.c is colored with ≤ 4 colors, constraint $cn(\Upsilon_i) \neq cs(\Upsilon_j)$ can always be satisfied.*

Proof. The proof is straightforward. In Figure 1.a, without losing generality, assume γ_i, γ_j are colored $1, 2$ respectively, hence we can set $cn(\Upsilon_i) = \{2,3,4\}$, $cn(\Upsilon_j) = \{1,3,4\}$ which are different. The second part can be proved similarly. In Figure 1.c, $|\Upsilon_j| = 1$, if the graph is colored with ≤ 4 colors, $color(v) \notin cn(\Upsilon_i)$, hence by Definition 3, there is $cn(\Upsilon_i) \neq cn(\Upsilon_j)$.

□

For convenience, from now, to a cluster Υ, the vertex γ is called Υ's cluster-vertex. γ has edges to all vertices $\in \Upsilon$.

As we have seen, cluster whose cardinality is 1 has special property. Below we prove some important properties of it.

Lemma 1. *If cluster $\Upsilon_j = \{v\}$, and $\Upsilon_j \subseteq \Upsilon_i$, then constraint $cn(\Upsilon_i) \neq cn(\Upsilon_j)$ can be transformed into $cn(\Upsilon_i) = cn(\Upsilon_j)$ and $cn(\Upsilon_j) \neq cn(\Upsilon_j')$ where $\Upsilon_j' = \Upsilon_j = \{v\}$.*

Proof. As in Figure 1.*d*, $\Upsilon_j = \{v_4\}$ and $\Upsilon_j \subseteq \Upsilon_i$. In order to satisfy constraint $cn(\Upsilon_i) \neq cn(\Upsilon_j)$, by Observation 2, it is sufficient to color Figure 1.*d* with ≤ 4 colors. Notice that, if add multi-edges between v_4 and γ_i, color assignment is not affected.

□

If we decompose a series of perimeter trace U into a set of clusters $\Upsilon = \{\Upsilon_1, \Upsilon_2, ..., \Upsilon_x\}$ by order *as them appearing on the perimeter trace.*

Definition 5. *In a planar graph $G(V,E)$, U is a perimeter trace, rules of collection-constraints of coloring on U are defined following:*

1. *a cluster is colored with colors from a collection;*
2. *two clusters have the same or different collections;*
3. *if Υ_i, Υ_j has constraints, then there is no constraints between Υ_x, Υ_y where $x \in (i,j)$ and $y \notin (i,j)$.*

Definition 6. *In a planar graph G, collection-constraints ct are defined on its perimeter trace, define constraint-graph G_{ct} by extending G and following:*

1. *every cluster has a cluster-vertex, and if $cn(\Upsilon_i) = cn(\Upsilon_j)$ the two clusters share a cluster-vertex;*
2. *if $cn(\Upsilon_i) \neq cn(\Upsilon_j)$, there is edge $e(\gamma_i, \gamma_j)$.*

Observation 3 *G_{ct} is a planar graph, $G_{ct} \setminus G$ is a outerplanar graph.*

Observation 4 *In planar graph $G(V,E)$, U is a perimeter trace, there is a cluster $\Upsilon = \{v_1, ..., v_x\}$ orderly appearing on U, if Υ's cluster-vertex γ is added, then $\{U \setminus \{\Upsilon \setminus \{v_1, v_x\}\}\} \cup \{\gamma\}$ is a new perimeter trace.*

Observation 5 *In the subgraph on $U \cup \gamma$, where γ are all the cluster-vertices, U is a perimeter trace.*

Definition 7. *We say Υ_i, Υ_j are relatable clusters, if there is no constraints between Υ_x, Υ_y where $x \in (i,j)$ and $y \notin (i,j)$.*

4.2 Division Constraints

Cluster-vertices(γ-vertices) naturally take an order according to that of their belonging clusters. Making use of this order, γ-vertices can be divided into sets of relatable clusters each of which is a K_3, K_2 or K_1 division.

Definition 8. *Respect to ct and its constraint-graph G, we say two divisions* D_1, D_2 *are relatable, if there are* $\gamma_1 \in D_1, \gamma_2 \in D_2$*, and after adding edge* $e(\gamma_1, \gamma_2)$*, the new graph is still a constraint-graph respect two* $ct' = ct \cup \{cn(\gamma_1) \neq cn(\gamma_2)\}$.

Since a *cluster* is corresponding to a γ-vertex, without causing confusion below sometimes we refer a division as the clusters that corresponding to γ-vertices that form the division.

Definition 9. *Rules of division-constraints are defined following:*

1. *a division has a set of* ≤ 3 *collections.*
2. *two relatable divisions have inconsistent sets of collections.*
3. *all clusters on a vertex belong to the same division.*
4. *if there is an edge* $e(v_1, v_2)$ *and the two divisions of clusters on* v_1, v_2 *are both* K_3 *division, they are two different divisions.*

Observation 6 *Assume* $\{v_1, v_2\}$ *are two vertices on a perimeter trace* U*, and collection-constraints* ct *are defined on* U*, in which no vertices between* v_1 *and* v_2 *are included by clusters. If there are two divisions* D_1, D_2 *of clusters on* $\{v_1\}$ *and* $\{v_2\}$ *respectively, and* D_1, D_2 *are relatable.*

Observation 7 *There is a color assignment of* G_{ct} *uses* ≤ 4 *colors, if and only if there is a color assignment cl of G using* ≤ 4 *colors and satisfying collection-constraints, and* $3, 4$ *cases of division-constraints.*

Proof. cl_{ct} can be used on G to be cl. cl satisfies all collection-constraints following Observation 2. If vertex v is colored with $\{1\}$, then cluster-vertices connected with $\{v\}$ are colored with $\{2, 3, 4\}$. Hence they can belong to the same division. If there is edge $e(v_1, v_2)$, and $\{v_1, v_2\}$ colored with $\{1, 2\}$ respectively, then two K_2 divisions of v_1, v_2 can be colored with $\{2, 3, 4\}$ and $\{1, 3, 4\}$ respectively, whose corresponding collections are inconsistent. So $3, 4$ cases of division-constraints are satisfied. The reverse can be proved similarly.

5 Properties of Valid Constraints

In this Section, we assume in a planar graph $G(V, E)$, there is a series S of perimeter trace U and S is decomposed to be clusters $\Upsilon = \{\Upsilon_1, \Upsilon_2, ..., \Upsilon_x\}$ as the order appearing on S, and a set of collection-constraints ct has been defined on Υ. Respect to ct, sets of division-constraints are defined. If $u \in \Upsilon_i \in cs$, after deleting u from G, in new graph $G' = G \backslash u$, by Proposition 1, $S' = \{U \backslash u\} \cup N(u)$

is a series of G''s new perimeter perimeter trace and Υ_i is split into two clusters $\Upsilon_{i1}, \Upsilon_{i2}$, and denote the new cluster $\Upsilon_{N(u)} = \{N(u)\}$. We can define collection-constraints ct' and division-constraints on S' by inheriting constraints of S. Note that clusters $\Upsilon_{i1}, \Upsilon_{i2}$ inherit all constraints on Υ_i.

Proposition 2. *In graph $G(V,E)$, suppose there are valid constraints ct on clusters $\Upsilon = \{\Upsilon_1, \Upsilon_2, ..., \Upsilon_x\}$ as the order appearing on perimeter trace, at most 3 clusters have constraints among them to make their collections different from each other.*

Proof. Assume $\{\Upsilon_i, \Upsilon_j, \Upsilon_k\} (i < j < k)$ have their collections from each other, by Definition 5, in ct there is no cluster can have constraints to be different with have constraints to be different with $\{\Upsilon_i, \Upsilon_j, \Upsilon_k\}$ simultaneously.

□

Figure 3.c gives an example of three clusters whose collections are different, in which $\gamma_i, \gamma_j, \gamma_k$ correspond $\Upsilon_i, \Upsilon_j, \Upsilon_k$ respectively referring Definition 6.

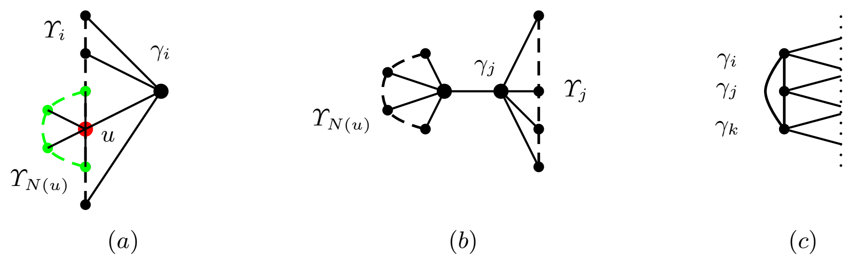


Fig. 3. Examples of Constraint Transformation

Lemma 2. *If $u \in \Upsilon_i \in \Upsilon$, after deleting u from G, in new graph $G' = G \setminus u$, Υ_i is split into two clusters $\Upsilon_{i1}, \Upsilon_{i2}$ and at least one of them is not empty, then we can add collection-constraints $cn(\Upsilon_{i1}) = cn(\Upsilon_{i2}) \neq cn(\Upsilon_{N(u)})$ into ct'. And if ct' are satisfied, then in ct Υ_i is colored with a collection.*

Proof. As shown in Figure 3.a, Υ_i is indicated by black dashed line, and $\Upsilon_{N(u)}$ green dashed line. By Observation 2, we get the conclusion immediately.

□

Lemma 3. *$\Upsilon_i = \{u\}$ and $u \notin \Upsilon_j$, after deleting u, if in ct $cn(\Upsilon_i) = cn(\Upsilon_j)$, then we can add collection-constraints $cn(\Upsilon_{N(u)}) \neq cn(\Upsilon_j)$ into ct'. And if ct' are satisfied, then in ct $cn(\Upsilon_i) = cn(\Upsilon_j)$ is satisfied.*

Proof. As shown in Figure 3.b, $\Upsilon_{N(u)}, \Upsilon_j$ are indicated by dashed line. By Observation 2, we get the conclusion immediately.

□

Easy to see Lemma 2 states a special condition of Lemma 3.

Cluster $\Upsilon_i = \{u\}$ can appear continuously because $|\Upsilon_i| = 1$, call them Υ_i clusters or u-clusters and denote such clusters $\{\Upsilon_{i1}, \Upsilon_{i2}, ..., \Upsilon_{ix}\}$. If we delete u, then those u-clusters disappear. As in Lemma 3, we also want to find a way to restore u and can satisfy constraints related with such u-clusters. For instance, if there is cluster Υ_j with $cn(\Upsilon_{i1}) \neq cn(\Upsilon_j)$, a natural way is to set $cn(\Upsilon_{N(u)}) = cn(\Upsilon_j)$. But if there is also Υ_k with $cn(\Upsilon_{i1}) \neq cn(\Upsilon_k)$ and $cn(\Upsilon_k) \neq cn(\Upsilon_j)$, we can not do the same thing to Υ_k, otherwise there is $cn(\Upsilon_k) = cn(\Upsilon_j)$ causing paradox. In next Subsection we will introduce how to add constraints into ct' and does break ct' valid property.

5.1 Properties of Constraints with Cardinality 1 Clusters

Recall that in a planar graph $G(V, E)$, on a perimeter trace U, there are clusters $\Upsilon = \{\Upsilon_1, \Upsilon_2, ..., \Upsilon_q\}$, and on such clusters, collection-constraints ct and division-constraints are defined. In G, there is vertex $u \in U$, if we delete u from G, we get a new graph $G' = G \setminus u$. In the new planar graph G', by Proposition 1, we have a new perimeter trace $U' = \{U \setminus u\} \cup N(u)$ on which there are clusters Υ'; in Υ', a new cluster $\Upsilon_{N(u)} = \{N(u)\}$ is added, and those clusters containing u are adjusted. For example, such clusters equal $\{u\}$ disappear. Also on Υ', use ct' to represent collection-constraints inheriting from ct.

Here we use cluster Υ_u to represent all clusters on $\{u\}$ which disappear because of deleting u. In Figure 4.a, Υ_{eq} represents clusters whose collections are the same with one cluster on $\{u\}$; Υ_{eq} represents clusters whose collections are different with one cluster on $\{u\}$. Notice that Υ_{eq} and Υ_{neq} can overlap with each other. γ_u, γ correspond to clusters including Υ_u and clusters in Υ_{neq} respectively. After deleting u, Υ_u is replaced by $\Upsilon_{N(u)}$. And if $u \in \Upsilon_4$, there is $cn(\Upsilon_{N(u)}) \neq cn(\Upsilon_4)$ displayed with dashed line in Figure 4.b.

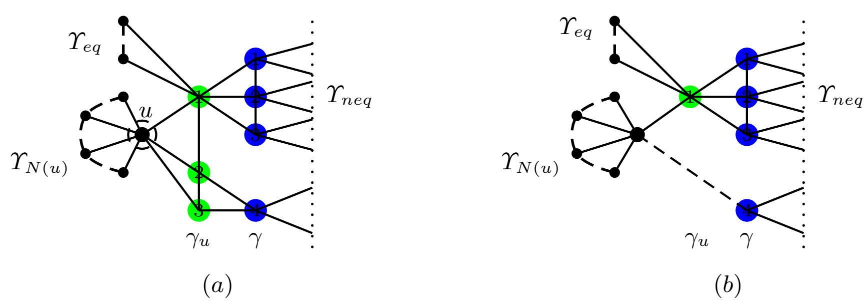


Fig. 4. Examples of Constraint Transformation

Lemma 4. *The collection-constraints and division-constraint of clusters including $\{u\}$ can be restored by defining extra constraints on related clusters on U' in graph G':*

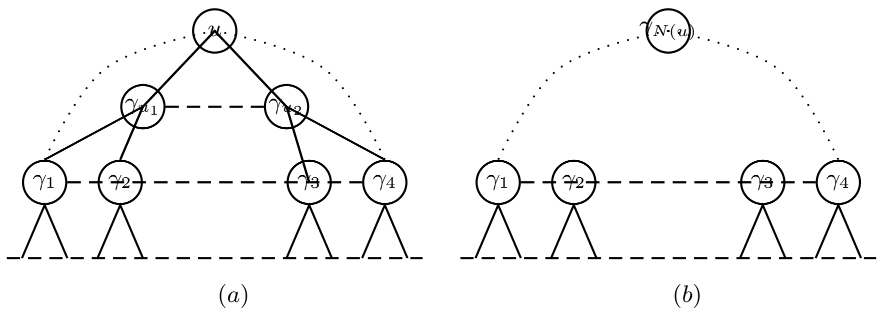


Fig. 5. Add New Constraints

1. *if there is $\Upsilon_i \neq \{u\}$ has collection-constraint $cn(\Upsilon_i) = cn(\Upsilon_u)$ where $\Upsilon_u = \{u\}$, then set $cn(\Upsilon_i) \neq cn(\Upsilon_{N(u)})$.*
2. *if there is $u \in \Upsilon_i$ and $\Upsilon_i \setminus \{u\} \neq \emptyset$, then cluster $\Upsilon_i' = \Upsilon \setminus \{u\}$ inherits all constraints on Υ_i.*
3. *assume D_u is the division of clusters on $\{u\}$, and there is a division D' in G'. If in G D_u, D' have consistent(inconsistent) sets of collections, then in G', make $\Upsilon_{N(u)}, D'$ belong to inconsistent(consistent) sets of collections.*
4. *if there is $\Upsilon_i = \{u\}$, then set cluster $\Upsilon_{N(u)}$ and all clusters with $cn \neq cn(\Upsilon_i)$ to belong a division, named D_i.*
5. *assume Υ_x, Υ_y are clusters on $\{u\}$, and belong to the same division in G_{ct}, then when 2 colors used to color the division $D_{i,j}$ formed of γ vertices between and include Υ_x, Υ_y, if:*
   (a) *γ_x, γ_y are used the same color, then divisions D_x, D_y have consistent collections.*
   (b) *γ_x, γ_y are used different colors, then divisions D_x, D_y have inconsistent collections.*

   Also set $D_{i,j}$ has inconsistent collections with D_x, D_y respectively.

Proof. If collection-constraints and division-constraints related with $\Upsilon_{N(u)}$ can be satisfied, assume $cn(\Upsilon_{N(u)} = \{1, 2, 3\}$, then cluster-vertex $\gamma_{N(u)}$, i.e. u, is colored with 4. In G, clusters on $\{u\}$ can have collections $\{1, 2, 4\}$, $\{1, 3, 4\}$ or $\{2, 3, 4\}$, whose corresponding cluster-vertices are colored with $\{3, 2, 1\}$ respectively.

1. if there is $\Upsilon_i \neq \{u\}$ has collection-constraint $cn(\Upsilon_i) = cn(\Upsilon_u)$ where $\Upsilon_u = \{u\}$, then set $cn(\Upsilon_i) \neq cn(\Upsilon_{N(u)})$.
   So assume $cn(\Upsilon_i) = \{1, 2, 3\}$, which is different from all possible collections of clusters on $\{u\}$.
2. if there is $u \in \Upsilon_i$ and $\Upsilon_i \setminus \{u\} \neq \emptyset$, then cluster $\Upsilon_i' = \Upsilon \setminus \{u\}$ inherits all constraints on Υ_i.
   Trivially if constraints on Υ_i' can be satisfied, so is constraints on Υ_i.
3. assume D_u is the division of clusters on $\{u\}$, and there is a division D' in G'. If in G D_u, D' have consistent(inconsistent) sets of collections, then in G', make $\Upsilon_{N(u)}, D'$ belong to inconsistent(consistent) sets of collections.

If D' has inconsistent collections with $\Upsilon_{N(u)}$, then its collections $\subseteq \{\{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$, which are consistent with D_u.
In another condition, if D' has consistent collections with $\Upsilon_{N(u)}$, then its collections include $\{1,2,3\}$, which are inconsistent with D_u.

4. if there is $\Upsilon_i = \{u\}$, then set cluster $\Upsilon_{N(u)}$ and all clusters with different collections, i.e. $cn \neq cn(\Upsilon_i)$, to belong a division, named D_i.
   Because D_i is a division, then it has ≤ 3 collections. So its γ vertices are colored with ≤ 3 colors including $\gamma_{N(u)} = \{u\}$. Hence γ_i can be colored with the used colors.
5. assume Υ_x, Υ_y are clusters on $\{u\}$, and belong to the same division in G_{ct}, then when 2 colors used to color the division $D_{i,j}$ formed of γ vertices between and include Υ_x, Υ_y, if:
   (a) γ_x, γ_y are used the same color, then divisions D_x, D_y have consistent collections.
   If D_x, D_y have consistent collections, then their cluster-vertices can have the same colors. Hence γ_x, γ_y are colored the same.
   (b) γ_x, γ_y are used different colors, then divisions D_x, D_y have inconsistent collections. Similar as above, γ_x, γ_y are colored with different colors.

   Also set $D_{i,j}$ has inconsistent collections with D_x, D_y respectively.
   If γ_x, γ_y are colored the same. Without losing generality, assume they are colored with $\{1\}$. I.e. D_x, D_y have cluster-vertices colored with $\{2,3,4\}$. So $D_{i,j}$ has cluster-vertices colored with colors including $\{1\}$, for convenience assume $\{1,2,3\}$. Hence γ_x, γ_y can be merged into $D_{i,j}$ to be a division in G. Also the cluster-vertices between γ_x and γ_y can be colored with $\{1,2\}$ or $\{1,3\}$ alternatively.
   If γ_x, γ_y are colored differently. Without losing generality, assume they are colored with $\{1\}$ and $\{2\}$ respectively. I.e. D_x, D_y have cluster-vertices colored with $\{2,3,4\}$ and $\{1,3,4\}$ respectively. So $D_{i,j}$ has cluster-vertices colored with colors including $\{1,2\}$, for convenience assume $\{1,2,3\}$. Hence γ_x, γ_y can be merged into $D_{i,j}$ to be a division in G. Also the cluster-vertices between γ_x and γ_y can be colored with $\{1,2\}$ alternatively.

To any 3-collection including $\{4\}$, if color $\{4\}$ is removed, the collection is reduced to be a 2-collection. From case 4, we can see that every division of clusters whose cluster-vertices having edges with cluster-vertices on a cluster of $\{u\}$ has its cluster-vertices colored with colors including $\{4\}$. By Theorem 3 in [**?**], all clusters of $\{u\}$ can have their cluster-vertices colored with $\{1,2,3\}$. Hence these clusters belong to the same division.

$\square$

6 To Prove Four Color Theorem

Theorem 1. *Given a planar graph $G(V,E)$ and its perimeter trace U, G has a color assignment cl using ≤ 4 colors when it satisfies all collection-constraints and division-constraints.*

Proof. We make induction on $|V|$. When $|V| = 1$, suppose $V = \{v\}$. The constraint-graph G_{ct} defined by collection-constraints ct have $\{v\}$ and all γ vertices colored with $\{1\}$ and $\{2, 3, 4\}$ respectively. Hence division-constraints can be verified as satisfied. Then by Observation 13, the conclusion holds when $|V| = 1$.

When $|V| = n$, after removing a vertex $u \in U$, the graph $G' = G \setminus u$ is a planar graph and by Proposition 1 $U' = \{U \setminus u\} \cup \{N(u)\}$ is included by a perimeter trace. By Lemma 4, G has a color assignment cl using ≤ 4 colors when it satisfies all collection-constraints and division-constraints.

□

Hence by Theorem 1, we have proved four color theorem, and conclude as a corollary as below.

Corollary 1. *Every planar graph is* 4 *colorable.*

7 Conclusion

The proof in this paper can be treated as a generalization of proof in [3]. In this paper, we have proved four color theorem, but properties of planar graph are utilized, hence can not be generalized to prove Hadwiger Conjecture. However in [7,**?**] a bunch of results can be used to prove condition Hadwiger Conjecture when $k = 5$ without using property of planar graph. *Hence, we claim the ideas and conclusions in this paper can be generalized to prove Hadwiger Conjecture.*

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An Equivalent Statement of Hadwiger Conjecture when $k = 5$

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1 Abstract

Hadwiger conjecture states that if a graph has no K_k minor, then its chromatic number is $k-1$. In this paper, we study hadwiger conjecture when $k = 5$ and give two new results. One is that in a chromatic number 5 graph a K_5 minor can be constructed otherwise the graph would be reduced by applying minor actions to be a minimum vertex degree ≥ 5 graph with no consistent cut set; further if there is a degree 5 vertex, its neighbors are on a five-sided polygon. The other is hadwiger conjecture when $k = 5$ is proved to be equivalent with that in a chromatic number 4 graph, there is a K_4 minor on its kernel vertices. The later result is interesting as it can be used to find existence of special structures and it can greatly simplify the proof of the case when $k = 4$ of hadwiger conjecture. All conclusions we give can be generalized to arbitrary k on hadwiger conjecture.

2 Introduction

In graph theory, the hadwiger conjecture [1] states that, if all colorings of an undirected graph G need $\geq k$ colors, then G has a K_k minor. When $k = 5$, this conjecture is equivalent with four color problem which states that every planar graph has a chromatic number 4 [6,3,4,5,6], i.e., a chromatic number 5 graph has a K_5 minor.

Four color theorem has been proved assisted by computer for the first time in 1976 by Kenneth Appel and Wolfgang Haken [7,8]. A simpler proof using the same idea and also relied on computer was given in 1997 by Robertson, Sanders, Seymour, and Thomas [9,10]. Additionally in 2005, the theorem was proved again by Georges Gonthier [11,12] with general purpose theorem proving software which is also relied on computer. All these proofs have one thing in common that they are all complicated computer-assisted proofs which render it unreadable and uncheckable by hand. Moreover, all of current computer-assisted proofs of four color theorem use unavoidability configures of planar graph, and this makes them difficult to be generalized to prove Hadwiger conjecture. In this sense, it is still important to do research on Hadwiger Conjecture.

In this paper, we study Hadwiger conjecture of case $k = 5$ directly and give a series of theoretical results, in the meanwhile we hope our results are easy to

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be generalized to any k of Hadwiger conjecture. At first we will show that a chromatic number 5 graph has a K_5 minor otherwise it is a graph which can be reduced by applying minor actions to be a smaller chromatic number ≥ 5 graph, which has no consistent cut set, and with minimum vertex degree ≥ 5. This conclusion is similar but different with Dirac's results collected in [5] by using critical graph. The knowledge of alternative path is not needed in our proof which makes the proof simpler. And in Figure 1, an example shows that our result can be used to reduce the graph but not Dirac's results. Further, some new interesting conclusions can be deduced like Proposition 1. All such contents are put in Section 4.

The other contribution is we prove finding a K_5 minor in a chromatic number 5 graph is equivalent to finding a K_4 minor on a set of special vertices of its chromatic number 4 subgraph. Because our proof does not depend on any property of planar graph, this result can be used to give a restatement of Hadwiger Conjecture. To get the new statement is not complicated, but by using the new statement it may simplify the proof of Hadwiger Conjecture. In order to show this, we reprove Hadwiger Conjecture when $k = 4$ as in Theorem 5. Also, by using the new statement, we can show existences of some simple structures like simple cycle and forest as stated in Theorem 4. These materials are discussed in Section 5.

3 Terminology and Definition

In this Section, necessary terminologies and definitions are introduced.

Definition 1. *A color assignment to a graph $G(V, E)$ is a set of partitions of V, in which each partition is an independent set and different partition is assigned with a different color.*

In this paper, we use cl to denote a color assignment and integers to represent colors. Then we can say there is a l-color assignment $cl = \{1, 2, ..., l\}, |cl| = l$. Given a graph $G = (V, E)$ and a color assignment cl. Function $color_{cl}(v)$ represents vertex v's color in cl. Without confusion we sometimes skip the subscript cl. Given a set of vertices $W \subset V$, $color(W)$ means all the colors used on W in a color assignment. To a color assignment cl, define its $frequency - vector$, abbreviated as fv, as $< times_{cl}(l), times_{cl}(l-1), ..., times_{cl}(1) >$, where $times$ means how many times a color is used in cl. Then we can compare two color assignments by their $frequency - vector$ in lexicographical order.

If color assignments are ordered by their $frequency-vectors$ in lexicographical order, there exists a color assignment with the minimum $frequency-vector$. Name this color assignment as CL and the corresponding $frequency - vector$ as NV. Below without explicit explanation, we always use the color assignment whose $frequency - vector$ is minimum.

Definition 2. *In chromatic number k graph $G(V, E)$, $U \subseteq V$, if in every color assignment of G using k colors, U are assigned with k colors, then U is called a set of kernel vertices of G.*

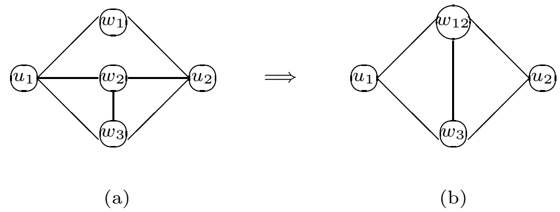

Fig. 1. Examples of Reductions According to Certain Admissive Relation

Trivially V is always a set of kernel vertices of $G(V, E)$.

Given a graph $G = (V, E)$, a set of vertices $S = \{s_1, s_2, ..., s_x\} \subset V$ and an equivalence relation R on S, we use $abs_R(S)$ or S_R to represent a new set of vertices, in which every vertex is contracted from an equivalent subset of S. A vertex belonging to S_R is called super-vertex of R from S. Then we can have a new graph $G_R = (V_R, E_R)$ by replacing S with $abs(S)$, and E_R is defined as below: if $s_i, s_j \in S$ are contracted to be s', then $N(s') = N(s_i) \cup N(s_j) \setminus \{s_i, s_j\}$; edges with no endpoint in S are intact.

In Figure 1 there is a example to show how to do contraction. In Figure 1.a, on the set of vertices $W = \{w_1, w_2, w_3\}$ define equivalence relation $R = \{\{w_1, w_2\}, \{w_3\}\}$; in Figure 1.$b$, the graph is G_R, in which w' is contracted from $\{w_1, w_2\}$. *One equivalent relation is called "admissive" if in the relation, v_1, v_2 are equivalent implies no edge $e(v_1, v_2)$ between them.* The relation in Figure 1 is admissive.

Observation 8 *After defining equivalence relation R on a set of vertices S, if abs is an admissive relation, then for every color assignment cl' of G_{abs}, cl' can be extended to be cl as a color assignment of G, where $color_{cl}(v) = color_{cl'}(v)$ for $v \in V \setminus S$, and $color_{cl}(S_i) = color_{cl'}(s_i)$.*

Proof. Since abs is admissive, then in S_i every two vertices are not connected by an edge. Hence we can set $color_{cl}(V \setminus S) = color_{cl'}(V \setminus S)$, and for every vertex $w \in S_i$, $color_{cl}(w) = color_{cl'}(s_i)$ directly.

Observation 9 *In a chromatic number k graph $G(V, E)$, $S \subseteq V$, and R is an admissive equivalence relation on S, then graph G_R has chromatic number $\geq k$.*

Proof. It follows from Observation 8 directly.

Definition 3. *In Graph $G(V, E)$ with cut set $W \subseteq V$ and admissive equivalence relation R on W, if in graph $G' = G \setminus W$, there are two subgraphs C_1, C_2 which are disconnected with each other, and in graphs $C_i \cup W (i \in \{1, 2\})$, by applying minor actions, $abs_R(W)$ can be achieved, and the subgraph on $abs_R(W)$ is a clique, then W is called a consistent cut set of G to R.*

Corresponding to a different minor action, we define an extension of a set of vertices as below.

Definition 4. *Given a graph $G(V, E)$ and a set of vertices $U \subseteq V$, if $G'(V', E')$ is reduced from G by applying a minor action, the extension of U, $U' \subseteq V'$ is defined as:i) if deleting a vertex $v_1 \in U$: $U' = \{U \setminus v_1\} \cup N(v_1)$; ii) if $v_1 \in U$ or $v_2 \in U$ and contracting $v_1, v_2 \in V$ with $e(v_1, v_2)$ to be v', $U' = \{U \setminus \{v_1, v_2\}\} \cup v'$; iii) otherwise, $U' = U$.*

When minor actions are applied sequentially, extension can be defined iteratively and an iterated extension is denoted as $Ex(U)$. U is trivially an extension of itself when no action applied.

Conventionally, given a graph $G = (V, E)$, a vertex $v \in V$, a set of vertices $W \subseteq V$, and a subgraph $G_s = (V_s, E_s)$ of G, a new subgraph $G'_s(V'_s, E'_s) = G_s \cup v$ means that $V'_s = V_s \cup v$ and $E'_s = E_s \cup \{edges\ between\ v\ and\ V_s\ in\ G\}$; a new subgraph $G'_s(V'_s, E'_s) = G_s \cup W$ means that $V'_s = V_s \cup W$ and $E'_s = E_s \cup \{edges\ between\ W\ and\ W \cup V_s\ in\ G\}$. G_W represents the subgraph $G \cap W$.

Definition 5. *In graph $G(V, E)$, $U \subseteq V$ and $|U| = x > 0$, we say in G there is a K_x minor on U under this condition: if a vertex $v \in K_x$ is contracted from $S \subseteq V$, then $U \cap S \neq \emptyset$. If $U' \subseteq V$ and $U \subseteq U'$, we also say there is a K_x minor on U'.*

Assume c_1, c_2 are two colors, define $f(c_1, c_2)$ as the two color exchange function by exchanging colors c_1 and c_2. cl_f is the new color assignment by applying f on the color assignment cl. For convenience, sometime we use f to represent one color exchange function instead of $f(c_1, c_2)$.

Observation 10 *If f is a color exchange function, then $f \circ f = e$; if $f = f_1 \circ f_2 \circ ... f_t$ and $f' = f_t \circ ... f_2 \circ f_1$, then $f \circ f' = e$. Here e means the identity function.*

Proof. The proof is straightforward.

4 A New Method To Reduce Graphs

In this section, we will show that a chromatic number 5 graph $G(V, E)$ can be reduced by minor actions to be a smaller graph whose chromatic number is ≥ 5, otherwise there is a vertex $v \in V$ such that $G' = G \setminus v$ is a 3-connected chromatic number 4 graph and $N(v)$ is G''s kernel vertices. In this paper, when we say a graph is n-connected, it means the graph is connected after removing arbitrary $(n-1)$ vertices.

A chromatic number 5 graph $G = (V, E)$ has a color assignment using only 5 colors. Name the set of vertices which are assigned color 5 as V_5. Define G' to be $G'(V', E') = G \setminus V_5$.

Lemma 1. *Given a chromatic number 5 graph $G = (V, E)$, and its minimum color assignment CL with a $frequency-vector$ NV, if $v \in V_5$, then graph $G \setminus \{V_5 \setminus \{v\}\}$ has chromatic number 5.*

Proof. Assume the new graph has chromatic number ≤ 4, then there exists a color assignment cl using ≤ 4 colors. W.L.O.G, assume colors $\{1, 2, 3, 4\}$ are used. Then in graph G, we can extend cl by coloring vertices $V_5 \setminus \{v\}$ with color 5. So in the new color assignment its $num(5)$ of the $frequency - vector$ is one less than $num(5)$ of NV. This is a contradiction with that NV is minimum.

By Lemma 1 we can always assume $|V_5| = 1$ in CL. Then if we have a vertex $v \in V_5$, then we can assume $V_5 = \{v\}$.

Lemma 2. *To the graph $G' = G \setminus V_5$ defined in Lemma 1, $N(v)$ is a set of kernel vertices.*

Proof. Assume there is one color assignment cl on G' using 4 colors $\{1, 2, 3, 4\}$ and $N(v)$ are colored with 3 colors $\{a, b, c\}$, then we can extend cl to be a 4-color assignment for graph $G' \cup v$ by assigning v with color $\{1, 2, 3, 4\} \setminus \{a, b, c\}$. It contradicts Lemma 1.

Lemma 3. *In graph $G(V, E)$, suppose W is a cut set of G and the subgraph $G_W = G \cap W$ is a clique; suppose C_l, C_2 are two subgraphs of graph $G' = G \setminus W$ which are disconnected with each other. If $G_l = C_l \cup W, G_r = C_l \cup W$ have color assignment cl_l, cl_r respectively, then G has a color assignment cl satisfying $color_{cl}(G) \leq max(color_{cl_l}(G_l), color_{cl_r}(G_r))$.*

Proof. Because the subgraph G_W is a clique, by color exchanging we can assume $color_{cl_l}(W) = color_{cl_r}(W)$. Hence cl_l, cl_r can be combined to be one color assignment for G without introducing more colors.

Lemma 4. *In graph $G(V, E)$, suppose W is a consistent cut set of G, the corresponding admissive equivalence relation on W is R, and the two subgraphs in $G' = G \setminus W$ are C_l, C_2. If $G_l = C_l \cup abs_R(W), G_r = C_l \cup abs_R(W)$ both have chromatic number $\leq k$, then G has chromatic number $\leq k$.*

Proof. Because G_l, G_r both have chromatic number $\leq k$, there are color assignments cl_l, cl_r using $\leq k$ colors for G_l, G_r respectively. By Definition 3, $abs_R(W)$ is a clique, hence we can set graph $GG = G_l \cup G_r$, and by Lemma 3, cl_l, cl_r can be combined to be one color assignment cl for GG using $\leq k$ colors. By Observation 8, cl can be extended to be one color assignment cl' for graph G, and cl' use $\leq k$ colors. So G has chromatic number $\leq k$.

Theorem 1. *Given a chromatic number 5 graph $G(V, E)$, suppose W is a cut set of G; setting $G' = G \setminus W$, assume C_l, C_r are two subgraphs of G' which are disconnected with each other. If W is a consistent cut set of G, then there is a K_5 minor in G, otherwise at least one of $G'_l = abs(W) \cup C_l, G'_r = abs(W) \cup C_r$ has its chromatic number ≥ 5.*

Proof. By Definition 3, $abs_R(W)$ is a clique. If $abs_R(W)$ is a ≥ 4 clique, a K_5 minor can be constructed immediately. If not, $abs_R(W)$ is a $i \leq 3$ clique. Suppose G'_l, G'_r both has chromatic number ≤ 4, by Lemma 4, G has its chromatic number ≤ 4 which is a contradiction with assumption. Hence, at least one of graphs G'_l, G'_r has its chromatic number ≥ 5.

Theorem 1 can be generalized to different k of Hadwiger Conjecture.

Corollary 1. *If graph $G = (V, E)$ has chromatic number 5, then G is at least 4-connected, otherwise G can be reduced by minor actions to be a smaller graph with chromatic number ≥ 5.*

Proof. By Theorem 1, we only need to show a cut set in G whose cardinality ≤ 3 is consistent. Assume in G there is a minimum cut set W with $|W| \leq 3$. It is easy to see that there always exists an admissive equivalence relation$R$ on W with $abs_R(W)$ as a clique. Assume C_l, C_r are two subgraphs of $G' = G \backslash W$ which are disconnected with each other. From simple analysis of cases depending on $G \cap W$, by doing minor actions on C_l, $abs_R(W)$ can be achieved; so is C_r. One case of $|W| = 3$ is displayed in Figure 1.a, in which $W = \{w_1, w_2, w_3\}$, and say $u \in C_l$. Because W is the minimum cut set in G, u can connect with W along three pathes showed as dash-line in the figure. There are two ways to define admissive relations on W and either one is sufficient to prove our conclusion:

1. $R = \{\{w_1, w_3\}, \{w_2\}\}$ By applying minor actions, w_1, w_3 can be contracted along pathes $P_{w_1,u}$ and P_{u,w_3} to be vertex w', then we get the $abs_R(W)$ as a 2-clique.
2. $R = \{\{w_1\}, \{w_2\}, \{w_3\}\}$. Similarly we can get the $abs_R(W)$ as a 3-clique.

All other cases can be analyzed similarly. Hence W is a consistent cut set of W.

Corollary 2. *In a chromatic number 5 graph $G = (V, E)$, if G has minimum vertex degree < 5 then it has a K_5 minor or G can be reduced by minor actions to be a smaller graph with chromatic number ≥ 5.*

Proof. By Corollary 1, G is at least 4-connected. So we only need to discuss the case when there is a vertex $v \in V$ with degree 4, then in this case $W = N(v) = (v_1, v_2, v_3, v_4)$ is a minimum cut set.

If $G \cap N(v)$ is a 4-clique, then $v \cup N(v)$ is a 5-clique and hence a K_5 minor. Otherwise, there is no K_5 minor in G, then W.L.O.G. we can assume there is no edge $e(v_1, v_2)$ where $v_1, v_2 \in N(v)$. Define the equivalence relation$R$ on $N(v)$ as: $\{\{v_1, v_2\}, v3, v4\}$.

Then we have $|abs_R(W)| < 4$ and by applying minor contractions only on $v \cup N(v)$, $abs_R(W)$ can be achieved. Suppose $G' = \{G \setminus \{v \cup N(v)\}\} \cup abs_R(W)$ has chromatic number < 5, then there is a color assignment cl' of G' using < 5 colors. Since $abs_R(W)$ can be colored with < 4 colors, hence cl' can be extended to be a color assignment of graph G_R using < 5 colors. By Observation 9, we can have a color assignment for G using < 5 colors which is a contradiction. Hence the chromatic number of G' has to be ≥ 5.

Similar results from Dirac as Corollary 1 and Corollary 2 are collected in [5] by working on critical graph, compared with original proofs, alternative path is not used in our proofs which makes our proofs simpler. And our conclusions are different with original results. For example, in Figure 1.b, assume $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ is a cut set and u_1, u_2 are in one component C_l in $G' =$

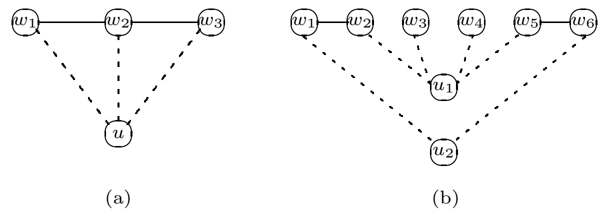

Fig. 2. Examples of Reductions According to Certain Admissive Relation

$G \setminus W$. With results of [5], this graph can not be contracted to be a smaller graph with chromatic number ≥ 5. But we can define an admissive relation R on W by putting $R = \{\{w_1, w_6\}, \{w_2, w_3, w_4, w_5\}\}$, then by contracting pathes passing through u_1 and u_2 displayed in dash-line in Figure 1.b, $abs_R(W)$ can be achieved as a 2-clique. If we can do the same thing in $G' \setminus C_l$, then by Theorem 1 G can be contracted to be a smaller graph G_R whose chromatic number ≥ 5. Further, we have that:

Proposition 1. *In a chromatic number* 5 *graph* $G = (V, E)$*, if* $v \in V$ *has degree* 5*, then* $N(v)$ *is a five-sided polygon, otherwise* G *has a* K_5 *minor or* G *can be reduced by applying minor actions to be a smaller graph with chromatic number* ≥ 5.

Proof. Assume $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$. By Corollary 1 $G' = G \setminus v$ is at least 3 connected. If there is a triangle on $N(v)$, assume on $\{v_1, v_2, v_3\}$, then from v_4(or v_5) to the triangle there are at least 3 disjoint paths. Hence on $\{v_1, v_2, v_3, v_4\}$ there is a K_4 minor without using v, then combined with v, there is a K_5 minor. If there is no triangle on $N(v)$, and by proof of Corollary 2, there is no three independent vertices in $N(v)$ otherwise G can be reduced, the subgraph on $N(v)$ can only be a five-sided polygon.

If the chromatic number 5 graph is a planar graph, it has its minimum degree ≤ 5. That means in four color problem, if there is a minimum counter example, i.e. a planar graph can not be colored with ≤ 4 colors, *in it there is a vertex* v *with* $deg(v) = 5$*, and* $N(v)$ *is on a five-sided polygon.* If $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ as the order appearing on the polygon, are colored with colors $\{1, 2, 3, 4\}$, easy to see $N(v)$ are colored with $\{1, 4, 2, 3, 4\}$ respectively. Hence, *it can be proved easily a planar graph is* 5*-colorable.*

Theorem 2. *Given a chromatic number* 5 *graph* $G = (V, E)$*, if* G *can not be reduced by applying minor actions to be a smaller graph with chromatic number* ≥ 5*, then there is a vertex* $v \in V$*, such that* $G' = G \setminus v$ *is 3-connected and* $N(v)$ *is a set of kernel vertices of* G'.

Proof. If G can not be reduced by applying minor actions to be a smaller graph with chromatic number ≥ 5, then we can choose a vertex $v \in V_5$, by Lemma 2 $N(v)$ is a set of kernel vertices of G'. By Corollary 1, G' must be at least 3-connected.

For convenience in rest of this paper, without explicit explanation, we assume a given chromatic number k graph can not be reduced to be a chromatic number $\geq k$ graph by applying minor actions. The assumption is reasonable, because if not we can work on the smaller graph.

5 A Restatement of Hadwiger Conjecture

In this section we conceive a new conjecture and we prove it is equivalent with the case $k = 5$ of Hadwiger Conjecture.

Conjecture 1. In a 3-connected chromatic number 4 and K_5 minor free graph $G(V, E)$, if U is a set of kernel vertices of G, then there is a K_4 minor on $Ex(U)$.

Hadwiger Conjecture when $k = 5$ claims the existence of K_5 minor in a chromatic number 5 graph. If a graph G has a subgraph containing a K_5 minor, then trivially there is a K_5 minor in original graph G. Below without explicit explanation it is assumed that the discussed graph $G(V, E)$ has all its subgraphs are K_5 minor free to eliminate this trivial case.

Theorem 3. *In a chromatic number* 5 *graph G, there exists a K_5 minor if and only if Conjecture 1 is correct.*

Proof. We need to prove that if Conjecture 1 is correct, then chromatic number 5 graph has one K_5 minor. The reverse can be proved similarly. Assume Conjecture 1 is correct. By Theorem 2, there is a vertex $v \in V$ satisfying $G' = G \setminus v$ is 3-connected and in G' $N(v)$ is a set of kernel vertices. By Conjecture 1, there is a K_4 minor on $Ex(N(v))$ in G'. Next we can show that the K_4 minor on $Ex(N(v))$ can be combined with v to be a K_5 minor in G. This can be done by applying minor actions in graph G so that $Ex(N(v))$ can be achieved as neighbors of v.

From Definition 3, the only non-trivial case comes from applying the action of deleting one vertex $v_1 \in Ex(N(v))$. Since at beginning it is initialized that $Ex(N(v)) = N(v)$, i.e. trivially $Ex(N(v))$ are neighbors of v in G. When we delete a vertex $v_1 \in Ex(N(v))$ in G', correspondingly in graph G we contract this vertex v_1 with v, hence $N(v_1)\backslash v$ become neighbors of v. By Definition 3, after deleting v_1 in G' the new $Ex(N(v))$ is updated to be $\{Ex(N(v)) \setminus v_1\} \cup N(v_1)$ which are v's new neighbors in graph G. Hence, the K_4 minor on $Ex(N(v))$ can be combined with v to be a K_5 minor.

Easy to see that, Theorem 3 can be generalized to give a new statement of Hadwiger Conjecture as below:

Corollary 3. *Hadwiger Conjecture when $k = x$ is correct if and only if in a chromatic number $(x - 1)$ graph, there is a K_{x-1} minor on its kernel vertices.*

In order to show the new statement of Hadwiger Conjecture is useful, at first we use it to prove some new properties, and then we give a quite simple proof to Hadwiger Conjecture when $k = 4$.

Lemma 5. *If graph $G(V,E)$ is 2-connected, then there is a K_3 minor on its arbitrary three vertices.*

Proof. Assume there are three vertices $\{v_1, v_2, v_3\} \subseteq V$, because G is 2-connected, then from v_1 to v_2 and form v_1 to v_3 there are two vertex disjoint pathes $P_{1,2}, P_{1,3}$ respectively.

Similarly, because G is 2-connected, from v_2 to v_3 there is one path $P_{2,3}$ which does not pass through v_1. Easy to see no matter how $P_{2,3}$ crosses with $P_{1,2}, P_{1,3}$, there is one K_3 minor on v_1, v_2, v_3.

Corollary 4. *If graph $G(V,E)$ is 3-connected, and there is a vertex $v \in V$ satisfying $|N(v)| \geq 3$, then there is a K_4 minor in $v \cup N(v)$.*

Proof. Because G is 3-connected, $G' = G \setminus v$ is 2-connected. Then by Lemma 5, there is a K_3 minor on $N(v)$. Hence, there is a K_4 minor in $v \cup N(v)$.

For convenience below we use @ to represent a simple cycle. Without confusion @ also represent the vertices on the cycle.

Lemma 6. *If graph $G(V,E)$ is 3-connected, $U \subseteq V$, and the sub-graph $U \cap G$ includes one simple cycle @ such that $U \setminus @ \neq \emptyset$, then there is a K_4 on U.*

Proof. Because $U \setminus @ \neq \emptyset$, assume $v \in U \setminus @$. Since G is 3-connected, v can connect with the simple cycle @ via three disjoint path. So $v \cup @$ can be reduced to be a K_4 minor.

Lemma 7. *If graph $G(V,E)$ is 3-connected, $U \subseteq V$, then*

1. *there is a K_4 minor on $Ex(U)$; or*
2. *the subgraph $U \cap G$ is a simple cycle or a reduced forest, in which every tree is a path graph.*

Proof. It follows from Corollary 4 and Lemma 6. Only need to notice that if a tree is not a path graph, then there is a vertex v with $|N(v)| \geq 3$.

Theorem 4. *In a chromatic number 5 graph $G(V,E)$ which can not be reduced to be a smaller graph by minor actions, choosing vertex $v \in V$ and setting $G' = G \setminus v$, then in G a K_5 minor can be constructed, otherwise $N(v) \cap G'$ is a simple cycle or a reduced forest, in which every tree is a path graph.*

Proof. By Theorem 2, we can assume G' is 3-connected. By Theorem 3, if there is a K_4 minor in $N(v)$, there is a K_5 minor in $v \cup N(v)$. Hence by Lemma 7 either a K_5 minor can be constructed, otherwise the subgraph $N(v) \cap G'$ is a simple cycle or a reduced forest, in which every tree is a path graph.

The Theorem below has been proved by Dirac in [2], and we reprove it with a much simpler proof to show why the new statement of Hadwiger Conjecture in Corollary 3 may be used to simply proof of Hadwiger Conjecture.

Theorem 5. *If graph $G(V,E)$ has chromatic number 4, there is one K_4 minor.*

Proof. Similar as Corollary 1, G can be assumed to be 3-connected. So choose one $v \in V$, there is $|N(v)| \geq 3$. By Corollary 4, there is a K_4 minor in $v \cup N(v)$.

Our proof of Theorem 5 is simple and it gives a way to find a K_4 minor which is not included in [2].

6 Conclusion and Next Step of Work

In this paper, we have shown that to a chromatic number 5 graph, if it can not be reduced to be a smaller chromatic ≥ 5 graph, then it has a K_5 minor, otherwise the graph has no consistent cut set, and its minimum vertex degree ≥ 5; further if there is a degree 5 vertex, for example in a planar graph, its neighbors are on a five-sided polygon. Moreover, we give a new statement of Hadwiger Conjecture. By working on the new statement, to find a K_5 minor in one chromatic number 5 graph is equivalent to find a K_4 minor on a chromatic number 4 subgraph's kernel vertices, and such kernel vertices have been proved to consist some special structures. Also, via the new statement, Hadwiger Conjecture when $k = 4$ can be proved easily and a K_4 minor can be constructed.

In next paper [8], we will strength the conclusion of Theorem 4 to show that the set of kernel vertices $N(v)$ locate on a simple cycle no matter $N(v) \cap G'$ is a simple cycle or a forest. Also we will show that by using the new statement of Hadwiger Conjecture, we can give a simple proof of Wagner's Equivalence Theorem without using Kuratowski's Theorem compared with existing proofs [6,3,4,5,6].

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A New Proof of Wagner's Equivalence Theorem

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1 Abstract

In this paper, we prove that in a 3-connected chromatic number 4 graph $G(V, E)$ given a vertex set $U \subseteq V$, if there is no K_4 minor on U, then U is included by a simple cycle of G. Moreover, when G is a planar graph, the simple cycle is boundary of an external face of G. Based on above results we give a new proof of Wagner's Equivalence Theorem without using Kuratowski's Theorem which is different from existing proofs.

2 Introduction

Hadwiger Conjecture [1] states that, if an undirected graph G has chromatic number k, then it has a K_k minor. When $k = 5$, this conjecture is a generalization of four-color problem. The four color theorem has been proved assisted by computer [7,8,9,10,11,12]. All such proofs have difficulties in readability and checkability, and can not be generalized to prove arbitrary k of Hadwiger Conjecture, hence it is still important to do research on the connection between Hadwiger Conjecture when $k = 5$ and four color problem in order to find the hidden properties which may lead to a short and generalizable proof to Hadwiger Conjecture.

In [7] it concludes that to prove Hadwiger Conjecture when $k = 5$ is equivalent to prove that a 3-connected chromatic number 4 graph $G(V, E)$ has a K_4 minor on its kernel vertices. In this paper, we prove that in a 3-connected chromatic number 4 graph $G(V, E)$ given $U \subseteq V$, either K_4 minor on U can be found or a subgraph of U has certain and elegant structures.

By above results, we find a way to apply induction method on graphs to prove properties of U. It aids us to give a new proof of Wagner's Equivalence Theorem without using Kuratowski's Theorem which is different from existing proofs [6,3,4,5,6].

In section 3, terminologies and some preliminary results are introduced. In section 4, it is shown that in a 3-connected chromatic number 4 graph $G(V, E)$, on a vertex set $U \subseteq V$ either a K_4 minor can be found, otherwise U is included by a simple cycle. In section 5, we give a new simple proof of Wagner's Equivalence Theorem.

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3 Terminology Definition and Preliminary Results

In this paper conventional graph theory terminology is applied and some definitions are quoted from [7].

Definition 1. *A color assignment to a graph $G(V, E)$ is a set of partitions of V, in which each partition is an independent set and different partition is assigned with a different color.*

We use cl to denote a color assignment and integers to represent colors. Then we can say there is a l-color assignment $cl = \{1, 2, ..., l\}, |cl| = l$.

Definition 2. *In a chromatic number k graph $G(V, E)$ given a vertex set $U \subseteq V$, if in every G's k-color assignment U are assigned with k colors, then U is called a set of kernel vertices of G.*

Corresponding to a different minor action, we define an extension of a set of vertices as below.

Definition 3. *Given a graph $G(V, E)$ and a set of vertices $U \subseteq V$, if $G'(V', E')$ is reduced from G by applying a minor action, the extension of U, $U' \subseteq V'$ is defined as:i) if deleting a vertex $v_1 \in U$: $U' = \{U \setminus v_1\} \cup N(v_1)$; ii) if $v_1 \in U$ or $v_2 \in U$ and contracting $v_1, v_2 \in V$ with $e(v_1, v_2)$ to be v', $U' = \{U \setminus \{v_1, v_2\}\} \cup v'$; iii) otherwise, $U' = U$.*

When minor actions are applied sequentially, extension can be defined iteratively and an iterated extension is denoted as $Ex(U)$. U is trivially an extension of itself when no action applied.

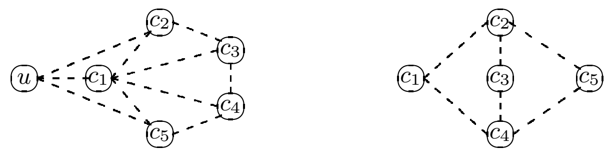
In a graph $G(V, E)$, given a vertex set $U \subseteq V$ and $|U| = x > 0$, we say in G there is a K_x minor on U under the following condition holds: if a vertex $v \in K_x$ is contracted from $S \subseteq V$, then $U \cap S \neq \emptyset$. If $U' \subseteq V$ and $U \subseteq U'$, we also say there is a K_x minor on U'.

In a simple cycle cy, after choosing arbitrary vertex $u \in cy$, starting with u, by tracing along the cycle cy, a *series* s of vertices is generated. After deleting vertices from s, the left series s' is still called a *series* of cy.

Two series s_1, s_2 are isomorphism if i) vertex $v \in s_1$ if and only if $v \in s_2$; and ii) by rotating or reversing the series, s_1 and s_2 can be transformed to be each other. Easy to see, *any two series corresponding to one cycle are isomorphic if and only if they contain the same set of vertices.* When we are talking series, if every series of a cycle has a special property, without confusion, we say the cycle has such a property.

A roundly continuous part of a series s is called a *cluster* of s. An empty set of vertices can be a cluster of any series. If we decompose s into a set of clusters $cs = \{cs_1, cs_2, ..., cs_x\}$ by order where $\bigcup_{i=1}^{x} cs_i = s$. cs is called clusters of series s if the end of cs_i may only overlap on ≤ 1 vertex with the beginning of $cs_{i+1 \ mod \ x}$.

For example, in Figure 1.a, we have cycle $cy = \{u, c_1, c_2, c_3, c_4, c_5, u\}$, then $s_1 =$ "c_2, c_3, c_4, c_5, u" and $s_2 =$ "u, c_2, c_3, c_4, c_5" are cy's two isomorphic series.



(a) (b)

Fig. 1. Examples of cycle and twin-cycle

In series s_1, because *one cluster can be chose roundly,* "u, c_2, c_3" *is one cluster.* {"c_3, c_4", "c_5, u", "u, c_2, c_3"} and {"c_3, c_4, c_5", "u", "u, c_2, c_3"} are two sets of clusters of s_1.

For convenience, to series s of cycle cy with clusters $cs = \{cs_1, cs_2, ..., cs_x\}$, we use cy_{cs_i} to represent the arc of cy containing cs_i and not containing the other clusters of cs. With two vertices $\{c_1, c_2\} \subset cy$, we use $cy_{c_1,c_2}, cy_{c_2,c_1}$ to represent the two split arcs of cy. Also if vertex $u \notin cy$, when we say u can connect with cy on a set of vertices $CR \subseteq cy$, that means u can connect with cy on CR without passing through the other vertices of cy. In Figure 1.a, set $cy = \{c_1, c_2, c_3, c_4, c_5, c_1\}$ and $u \notin cy$, then we say u can connect with cy on vertices $\{c_1, c_2, c_5\}$. In a simple cycle cy, when we say along the cycle $U = \{u_1, u_2, ..., u_x\} \subseteq cy$, that means vertices in U are in the order as they appear on the cycle.

Lemma 1. *In a graph $G(V,E)$, given a vertex $u \in U \subseteq V$ and a cycle cy satisfying that $u \notin cy$ and $|\{U\} \cap cy| \geq 3$ and u can connect cy on a set of vertices CR with $|CR| \geq 3$, if there is no K_4 minor on U, then CR and $\{U \cap cy\}$ are two clusters in any series including $CR \cup U$ on cy.*

Proof. The proof is straightforward.

Definition 4. *In a graph $G(V,E)$, given a vertex st $U \subseteq V$, we say a G's subgraph $G_s(V_s, E_s)$ is a $twin-cycle$ on U, if $U \subseteq V_s$ and:*

1. *there are two cycles cy_1, cy_2 with $cy_1 \cup cy_2 = G_s$;*
2. *$P = cy_1 \cap cy_2$ is one path with v_1, v_2 as the two endpoints;*
3. *$U \cap \{cy_1 \setminus P\} \neq \emptyset, U \cap \{cy_2 \setminus P\} \neq \emptyset$ and $U \cap \{P \setminus \{v_1, v_2\}\} \neq \emptyset$.*

We say the two vertices v_1, v_2 are crossing vertices of the $twin-cycle$, and $\{cy_1 \setminus P\} \cup \{v_1, v_2\}, \{cy_2 \setminus P\} \cup \{v_1, v_2\}, P$ are its three half-cycles.

Here we emphasize when there is one sub-graph G_s on U, without explicit explanation, it is assumed $U \subseteq cy$. In Figure 1.b, there is a twin-cycle on $\{c_1, c_3, c_5\}$ in which $P =$ "c_2, c_3, c_4" and $\{c_2, c_4\}$ are its two crossing vertices, and "c_2, c_1, c_4", "c_2, c_5, c_4", P are the three half-cycles.

We will use a simple operation named "reforming" on simple cycle and $twin-cycle$. Now we explain it with an example on Figure 1.a. In Figure 1.a, using the simple cycle $cy =$ "$c_1, c_2, c_3, c_4, c_5, c_1$", a set of vertices $U = \{u, c_2, c_4\}$, and

$I = U \cap cy = \{c_2, c_4\}$, then $u \in U$ and $u \notin I$. *We use reforming on cy to get a new cycle including* U. u connects with cy via two vertex disjoint pathes P_{u,c_2}, P_{u,c_5} which are terminating at cy at vertices c_2, c_5 respectively. Also name the path $P =$ "c_2, c_1, c_5" between c_2, c_5 along the cycle cy, easy to see $\{P \setminus \{c_1, c_2\}\} \cap U = \emptyset$, then we can reform cy by replacing $P \setminus \{c_1, c_2\}$ with $P_{u,c_2} \cup P_{u,c_5}$. After reformation, we have a new simple cycle $cy' =$ "u, c_2, c_3, c_4, c_5, u" and $cy \cap U \subset cy' \cap U$. We call this operation as the reforming of cy with respect to U.

Sometimes we will say to reform a simple cycle with respect to U, or *apply reforming* for abbreviation. Similarly, on $twin-cycle$, if c_1, c_2 are on a half cycle of the $twin - cycle$, we can define the same operation.

Lemma 2. *In a 3-connected graph $G(V, E)$ given a vertex set $U \subseteq V$ and $|U| \geq 4$, if there is a $twin - cycle$ on U, then there is a K_4 minor on U.*

Proof. Suppose there is $twin - cycle$ T on U with crossing vertices $\{a, b\}$, and three half cycles are cy_1, cy_2, cy_3. For convenience, set $cy'_i = cy_i \setminus \{a, b\}(i \in \{1, 2, 3\})$. By Definition 4, $U \cap cy'_i \neq \emptyset(i = \{1, 2, 3\})$, so assume $u_i \in cy'_i(i = \{1, 2, 3\})$. Because $|U| \geq 4$, we have cases as below:

1. there is $cy_i, i \in \{1, 2, 3\}$ with $|cy_i \cap U| > 1$: I.e. $|U \cap T| \geq 4$. assume $i = 1$, then in cycle $cy = cy_1 \cup cy_2$, there is $|U \cap cy| \geq 3$. Because G is 3-connected, u_3 can connect with cy without passing through $\{a, b\}$. I.e., u_3 can connect with cy at $CR = \{a, b, c\}$. Easy to see, no matter $c \in cy_1$ or $c \in cy_2$, CR and $\{U \cap cy\} \setminus CR$ can not be two clusters of any series of cy. By Lemma 1, there is a K_4 minor on U.
2. $|cy_i \cap U| = 1(i \in \{1, 2, 3\})$: I.e. $|U \cap T| = 3$. Because $|U| \geq 4$, there is $u_4 \in U$ and $u_4 \notin T$. Because G is 3-connected, from u_4 to T there are three vertex disjoint pathes p_1, p_2, p_3 crossing with $twin-cycle$ at $\{c_1, c_2, c_3\}$ respectively.
   (a) c_1, c_2, c_3 belong to one half cycle: assume $i = 1$. And the order along cy_1 is "a, c_1, c_2, c_3, b":
      i. u_1 is between c_1, c_3: by contracting cy'_2 with a to be u'_2, in cycle $cy = cy_1 \cup cy_3$ there is $|cy \cap U| = 3$ in which $\{u_1, u'_2, u_3\}$ and $\{c_1, c_2, c_3\}$ are not two clusters in any series of cy. By Lemma 1, there is one k_4 minor on U.
      ii. u_1 is not between c_1, c_3: assume the order is "a, u_1, c_1, c_2, c_3, b", then we can get one new $twin - cycle$ T' by reforming the $twin - cycle$ with respect to U. Then we have $|T' \cap U| \geq 4$ and can be analyzed as Case 1.
   (b) c_1, c_2, c_3 belong to two half cycles: assume $c_1, c_2 \in cy_1$ and $c_3 \in cy_2$. By contracting cy'_2 with a or b and Lemma 1, a k_4 minor on U can be constructed as in Case 2(a)i.
   (c) c_1, c_2, c_3 belong to three half cycles: this Case is similar as Case 2(a)i.

Lemma 3. *In a graph $G(V, E)$ given a vertex $v \in V$, if there is a twin-cycle on $N(v)$ then G has a $K_{3,3}$ minor.*

Proof. Assume $\{v_1, v_2, v_3\} \subseteq N(v)$ belong to the three half-cycles c_1, c_2, c_3 of one twin-cycle respectively. And a, b are the crossing points of the twin-cycle. Then on $\{v, a, b, v_1, v_2, v_3\}$ there is a $K_{3,3}$ minor.

Results below are needed in this paper. When we say a graph is n-connected, it means the graph is connected after removing arbitrary $(n-1)$ vertices.

Theorem 1. *[5,6] In a 4-connected graph $G(V, E)$, if there is a $K_{3,3}$ minor then there is a K_5 minor.*

Theorem 2. *[7] Given a chromatic number 5 graph $G = (V, E)$, if G can not be reduced by applying minor actions to be a smaller graph with a chromatic number ≥ 5, then there is a vertex $v \in V$, such that $G' = G \setminus v$ is 3-connected and $N(v)$ is a set of kernel vertices of G'.*

For convenience, call G, G' parent and child graph respectively. Easy to see $|N(v)| \geq 4$. It has been shown that an extension in G' can be got by applying minor actions in G which is not complicated to prove [7].

Conjecture 1. [7] In a 3-connected chromatic number 4 and K_5 minor free graph $G(V, E)$, if U is a set of kernel vertices of G, then there is a K_4 minor on $Ex(U)$.

Theorem 3. *[7] In a chromatic number 5 graph G, there exists a K_5 minor if and only if Conjecture 1 is correct.*

Proposition 1. *In a child graph $G'(V', E')$, there is no $K_{3,3}$ minor, otherwise G''s parent graph G has a K_5 minor.*

Proof. If there is a $K_{3,3}$ minor in G', then this $K_{3,3}$ minor exists in G. By Theorem 1, there is a K_5 minor in G.

4 Simple Cycle

In this section, we will show some interesting properties of a child graph G.

Lemma 4. *In a graph $G(V, E)$ given a vertex set $U \subseteq V$ and $|U| \geq 4$ which has no K_4 minor and contained in a cycle cy, if along the cycle there are $\{u_1, u_2, u_3, u_4\} \subseteq U$, then any path $P_{1,3}$ between u_1, u_3 crosses with any path $p_{2,4}$ between u_2, u_4 in graph G.*

Proof. The proof is straightforward.

Lemma 5. *In a graph $G(V, E)$ given a vertex set $U \subseteq V$ and $|U| \geq 4$, if U is contained in a cycle cy and has no K_4 minor, then isomorphically there is one unique series on U.*

Proof. At first, we can define one a s of U on cy. Suppose U has another series s' which is different from s, isomorphically s, s' have at least four vertices with different order, assume which are $\{v_1, v_2, v_3, v_4\}$, and assume in s the order is "v_1, v_2, v_3, v_4". Because $\{v_1, v_2, v_3, v_4\}$ are on cycle cy, if there is no K_4 minor on U, by Lemma 4, every path $P_{1,3}$ between v_1, v_3 crosses with one arbitrary path $P_{2,4}$ between v_2, v_4. So in s', beginning at v_1, the order of $\{v_1, v_2, v_3, v_4\}$ can only be "v_1, v_2, v_3, v_4'' or "v_1, v_4, v_3, v_2'', which are isomorphic. And this is a contradiction with assumption.

By Lemma 5, for a certain set of vertices U, *if there is no K_4 minor on U, we do not distinguish a cycle cy with $U \subseteq cy$ from the series of U in G.*

Theorem 4. *In 3-connected graph $G(V, E)$ given a vertex set $U \subseteq V$ and $|U| \geq 4$, if there is no K_4 minor on U then there is a simple cycle containing U.*

Proof. Choose $u_1, u_2 \in U$, because G is 3-connected, there is one cycle cy including u_1, u_2, i.e. we can assume $|cy \cap U| \geq 2$. If there is $u \in U$ and $u \notin cy$, because G is 3-connected, u can connect with cl on a set of vertices CR and $|CR| \geq 3$. Assume there is $CR = \{c_1, c_2, c_3\}$ and ordered as c_1, c_2, c_3 along cy. u can connect with $\{c_1, c_2, c_3\}$ via disjoint pathes P_1, P_2, P_3 respectively. According to $|cy \cap U|$, we have cases as below:

1. $|cy \cap U| = 2$: If $\{cy \cap U\}$ and CR are two clusters of a series of cy, cy can be reformed to be a new cycle and includes $\{u_1, u_2, u\}$ simultaneously, then the condition can be analyzed as the ≥ 3 case.

   If $\{cy \cap U\}$ and CR are not two clusters of any series of cy, then we can assume $u_1 \in cy_{c_1,c_2} \setminus \{c_1, c_2\}$ and $u_2 \notin cy_{c_1,c_2}$. Then there is one $twin-cycle$ with c_1, c_2 as the crossing vertices on $\{u, u_1, u_2\}$. Then by Lemma 2, there is one K_4 minor on U, which is a contradiction.
2. ≥ 3: By Lemma 1, CR and $\{U \cap cy\}$ are two clusters of a series of cy. So we can reform cy with respect to U. I.e., one more vertex in U can be included and no other vertices in U excluded. So iteratively, cy can be reformed to be a cycle containing all vertices of U.

Proposition 2. *In a 2-connected graph $G(V, E)$ given a vertex set $U \subseteq V$ which has no K_4 but is contained in a cycle cy, then for a vertex $u \in U$ and a vertex $u' \in N(u) \cap U \neq \emptyset$, if $U \cap \{cy_{u,u'} \setminus \{u, u'\}\} \neq \emptyset$ and $U \cap \{cy_{u',u} \setminus \{u, u'\}\} \neq \emptyset$, then $\{u, u'\}$ is a cut set of G.*

Proof. Assume $u_1 \in cy_{u',u}, u_2 \in cy_{u,u'}$. By Lemma 4, if there is no K_4 minor on U, every path between u_1, u_2 crosses with arbitrary a path between u, u'. Because there is edge $e(u, u')$ which is one path between u, u', any path can only cross with it via u or u'. Hence u, u' is a cut set of G.

When U is included by a cycle in 2-connected graph, Proposition 2 describes the structure of U. Below Proposition 3 and 4 describe structures of $Ex(U)$ when a vertex $u \in U$ is deleted from a 3-connected graph G.

Lemma 6. *In a 3-connected graph $G(V, E)$ given a vertex set $U \subseteq V$, if there exists a vertex $u \in U$ with $|N(u) \cap U| \geq 3$, there is a K_4 minor on U.*

Proof. Suppose there is such a vertex u with $\{u_1, u_2, u_3\} \subseteq \{N(u) \cap U\}$. By Theorem 4, $\{u, u_1, u_2, u_3\}$ are included by a simple cycle. Then by Proposition 2, there is a K_4 minor on U.

Proposition 3. *In 3-connected graph $G(V, E)$, $U \subseteq V$, $|U| \geq 4$, $u \in U$, if there is no K_4 minor on $Ex(U)$, then in graph $G' = G \setminus u$, $U' = \{U \setminus u\} \cup N(u)$, if G' is 3-connected, in G' there is no $twin - cycle$ on U'; and there is one simple cycle cy with $U' \subseteq cy$.*

Proof. By Lemma 6, $|N(u) \cap U| \leq 2$, so $|N(u) \setminus U| \geq 1$. Hence there is $|U'| \geq 4$. By Definition 3, U' is a extension of U. By assumption, there is no K_4 minor on U'. If G' is 3-connected, by Lemma 2, there is no $twin - cycle$ on U'; by Lemma 4, there is one simple cycle cy in G' with $U' \subseteq cy$.

Proposition 4. *In a 3-connected graph $G(V, E)$ given a vertex set $U \subseteq V$ with $|U| \geq 4$ and a vertex $u \in U$, set a graph $G' = G \setminus u$ and $U' = \{U \setminus u\} \cup N(u)$ when there is no K_4 minor on $Ex(U)$. If G' is 2-connected, then on U' there is a simple cycle C satisfying:*

1. *$\{U \setminus u\} \subseteq C$.*
2. *if $u_1 \in U' \setminus U$, then $u_1 \in C$ otherwise u_1 connects with C at exact two vertices. The two vertices form a cut set of G' by which $\{u_1\}$ and $\{U \setminus u\}$ are isolated.*

Proof. By Definition 3, U' is a extension of U, so there is no K_4 on U'. If there is a $twin - cycle$ on $U \setminus u$ in G', then this $twin - cycle$ exists in G on U, and by Lemma 2, there is one K_4 minor on U which is a contradiction. Hence there is no $twin - cycle$ on $U \setminus u$ in G'.

Because G' is 2-connected, by using reforming method, easy to prove there is one K_3 division on $U \setminus u$. Assume the K_3 division includes vertices $\{v_1, v_2, v_3\} \subseteq \{U \setminus u\}$. If the division is not a cycle, then we assume the division has a cycle cy on v_1, v_2, v and v_3 connects with the cycle at v. Because G' is 2-connected, v_3 can connect with the cycle at a different vertex v'. If $v' \in cy_{v_1,v} \setminus v$ on the division, then we can form a cycle along $v_1, v', v_3, v, v_2, v_1$; the same for $v' \in P_{v,v_2} \setminus v$. If $v' \in P_{v_1,v_2} \setminus \{v_1, v_2\}$, then treat v, v' as the two crossing vertices, there is one $twin - cycle$ with three half-cycles on $\{v_1, v_2, v_3\}$ respectively, which is a contradiction with no $twin - cycle$ on $U \setminus u$. Hence we can assume there is a cycle on $\{v_1, v_2, v_3\}$. Similar as proof of Lemma 4, we can reform to get a cycle C including all vertices of $U \setminus u$.

If there is $u_1 \in U' \setminus U$, because G' is 2-connected, there are two cases:

1. u_1 connects with C at ≥ 3 vertices: assume at W. Then by Lemma 1, W and $\{U' \cap C\}$ are two clusters. Then C can be reformed to contain u_1 and do not exclude any vertex in $U' \cap C$ out.

2. u_1 connects with C at 2 vertices: assume at $W = \{c_1, c_2\}$. Then $\{c_1, c_2\}$ is a cut-set of G' and in graph $G' \setminus \{c_1, c_2\}$, u_1 and $U \setminus u$ are in different components.

Proposition 5 show while keeping K_4 minor, a 2-connected graph can be reduced by applying minor actions.

Proposition 5. *In a 2-connected graph $G(V, E)$, let $\{v_1, v_2\}$ be a cut-set and in $G' = G \setminus \{v_1, v_2\}$ $C_1, C_2, ..., C_x$ be components. If G has a K_4 minor, then there is a K_4 minor on $C_i \cup \{v_1, v_2\}, i \in \{1, 2, ..., x\}$.*

Proof. Because $\{v_1, v_2\}$ is cut-set, every K_4 minor is on at most two components. Assume there is a K_4 minor on vertices $\{c_1, c_2, c_3, c_4\} \subseteq C_1 \cup C_2 \cup \{v_1, v_2\}$, then the only possibility is $c_1 \in C_1$ and $\{c_2, c_3, c_4\} \subseteq C_2 \cup \{v_1, v_2\}$, and $\{v_1, v_2\} \setminus \{c_2, c_3, c_4\} \neq \emptyset$. In order to get a K_4 minor, c_1 must be contracted with v_1 or v_2, hence a K_4 minor on $C_2 \cup \{v_1, v_2\}$ can be constructed.

5 Wagner's Equivalence Theorem

The results in section 4 can be applied in induction method to prove more interesting and useful results. Next as an exercise we will show how to use this induction to prove Wagner's Equivalence Theorem. Note that our proof does not depend on Kuratowski's Theorem.

Definition 5. *In a graph $G(V, E)$ given a non-empty vertex set $U \subseteq V$, we call G is the U's formal graph, if there is no K_5 or $K_{3,3}$ minor after adding a vertex v to G with $N(v) = U$.*

Lemma 7. *If a graph $G(V, E)$ is the formal graph of $U \neq \emptyset$, given a vertex $u \in U$ and U' the extension of U in graph $G' = G \setminus u$, then G' is a formal graph of U'.*

Proof. The non-trivial condition is $U' = \{U \setminus u\} \cup N(u)$. Suppose after adding v' to G' to get a new graph G'_v, there is K_5 or $K_{3,3}$ minor. Then in graph G, if we add v with $N(v) = U$, then contract v with u to be vertex v', the new graph is G'_v, hence there is K_5 or $K_{3,3}$ minor which is a contradiction with assumption.

Lemma 8. *In a graph $G(V, E)$ given a vertex set $U \subseteq V$, if G is the U's formal graph then there is no $twin - cycle$ on U.*

Proof. If there is a $twin - cycle$ T on U, after adding v to G with $N(v) = U$, by Lemma 3, there is a $K_{3,3}$ minor, which is a contradiction with Definition 5.

Theorem 5. *In a connected graph $G(V, E)$ given a vertex set $U \subseteq V$, if G is U's formal graph, then G is planar and U is contained by G's boundary of an external face.*

Proof. We prove this by induction on $|V|$. When $|V| \leq 3$, easy to verify the conclusion holds.

When $|V| > 3$, there are three cases:

1. G is 3-connected: $u \in U$, set $G'(V', E') = G \setminus u$, $U' = \{U \setminus u\} \cup N(u) \neq \emptyset$ is an extension of U. By Lemma 7, G' is a formal graph of U'. By induction, G' is planar, and U' is on boundary of an external face of G'. If restore u, G is still planar. And $N(u)$ and $U \setminus N(u)$ are two clusters, otherwise there is one $twin - cycle$ on U in G, which is a contradiction with Lemma 8. Hence, in G, U is on boundary of an external face.
2. G is 2-connected: Assume $W = \{w_1, w_2\}$ is an arbitrary cut-set of G, and C_1 is a component in graph $G \setminus W$. In graph $G_1 = C_1 \cup W$, we can assume there is edge $e(w_1, w_2)$, because otherwise we can apply contraction(minor action) on $G \setminus \{C_1 \cup W\}$ to contract W to be a vertex, and the proof is similar. Then graph G_1 has $U_1 = \{U \cap C_1\} \cup \{W\}$ as an extension of U by Definition 3. Similar as proof of Lemma 7, G_1 is a formal graph of U_1.

   By induction G_1 is planar and U_1 is on boundary of an external face. If in graph $G \setminus W$, there is another component C_2 and set $G_2 = C_2 \cup W$, then G_1, G_2 can be combined together by merging W, and after combination, it is planar and $U_1 \cup U_2$ is on boundary of an external face. If in graph $G \setminus W$, there are only components C_1, C_2, then this subcase has been proved.

   If in graph $G \setminus W$ besides C_1, C_2, there is another component C_3, then in G there is one $twin - cycle$ on U, which is a contradiction with Lemma 8. Hence we can conclude that when G is 2-connected, the conclusion holds.
3. G is 1-connected: This can be proved similarly as 2-connected case.

Theorem 5 has closed connection with Theorem 4, Proposition 3 and 4. Easy to see if condition of $K_{3,3}$ minor added, from results of Theorem 4, Proposition 3 and 4, we can prove Theorem 5 easily.

From Lemma 7 and Theorem 5, by intuition in a planar graph, the external face can be peeled iteratively. A reverse processing can be used to generated a planar graph. From these ideas, we can give a new geometrical definition of planar graph. With such a definition, simple algorithms can be designed to test planarity of graph and compute an orthogonal planar embedding of planar graph. Because of limit space, all of these will be discussed in another paper.

Lemma 9. *If $G(V, E)$ is a 4-connected chromatic number 5 and K_5 minor free graph, then graph G is planar.*

Proof. There is graph $G' = G \setminus v$ where $v \in V$. By Theorem 1 and Proposition 1, G' is a formal graph of $N(v)$. By Theorem 5, G' is planar, and $N(v)$ is on boundary of an external face of G'. Hence G is planar.

Lemma 10. *If each planar graph is 4-colorable, then Conjecture 1 is correct.*

Proof. We prove this Lemma by contradiction. Suppose Conjecture 1 is not correct, then by Theorem 3, there is a chromatic number 5 graph G in which there is no K_5 minor. By Theorem 2, G is 4-connected. By Lemma 9, G is planar.

By assumption, G can be colored with 4 colors which is a contradiction with G has chromatic number 5.

Theorem 6 (Wagner's Equivalence Theorem). *Every chromatic number* 5 *graph has a* K_5 *minor if and only if every planar graph can be colored with* 4 *colors.*

Proof. At first we prove from left to right. If a graph has a K_5 minor, trivially it is not planar. Suppose graph $G(V, E)$ is a planar graph, so G has no K_5 minor, by left side, G has chromatic number < 5. Hence G can be colored with 4 colors.

Then we prove from right to left. If every planar graph is 4-colorable, by Lemma 10, Conjecture 1 is correct. By Theorem 3, every chromatic number 5 graph has a K_5 minor.

6 Conclusion and Next Step of Work

In this paper, we prove that in a 3-connected chromatic number 4 graph $G(V, E)$ given a vertex set $U \subseteq V$, if there is no K_4 minor on U then U is included by a simple cycle of G. And when G is a planar graph, the simple cycle is boundary of an external face. By applying such results, we can prove Wagner's Equivalence Theorem without using Kuratowski's Theorem which is different from existing proofs. That means our proof does not rely on current existing properties of planar graph.

In fact starting from this paper a new geometrical definition of planar graph can be deduced, so is an algorithm for testing planarity and computing an orthogonal planar embedding of a planar graph whose complexity is the same as current algorithms but much simpler. In next step we will prove the new definition is equivalent with Kuratowski's Theorem.

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A Geometric View of Outerplanar Graph

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1 Abstract

In this paper, we show that an outerplanar graph can be drawn as a wing-1 graph whose vertices are drawn on a straight line and edges in one side along the line without edge crossing. This conclusion deduces a transformation drawing algorithm and gives us a clearer geometric view of an outerplanar graph to make better observation. The application of the conclusion helps us find new properties of 3-color assignments of outerplanar graphs. We also discuss a possible way to extend our results to color a planar graph with 4 colors.

2 Introduction

An outerplanar graph [1] is an undirected graph that can be drawn without edge crossing and whose vertices are on boundary of the drawing's unbounded or outer face. A graph is outerplanar if it is turned into a planar graph after adding a new vertex which connect all vertices in the graph. One method to recognize an outerplanar graph is to use its criterion: a graph is outerplanar if and only if it does not contain K_4 or $K_{2,3}$ minor [3]. The decomposition method to test if every biconnected component is outerplanar [2] can also be used.

All the knowledge in literatures on coloring an outerplanar graph is that it can be colored with 3 colors since its minimum vertex degree is no bigger than 2 [4]. A simple iterative algorithm can output a 3-color assignment which removes a degree ≤ 2 vertex, then colors the remaining graph, at last restores the removed vertex with the unused color different from colors assigned to its neighbors.

In this paper, we find a kind of graphs and name them as wing graphs. They have concise geometric view and good categorized property. A wing-1 graph is drawn as all its vertices on a straight line and edges on one side of the line without edge crossing. We prove an outerplanar graph is equivalent with a wing-1 graph. In the proof of the equivalence, a method can be deduced to draw an outplanar graph to a wing-1 graph which can also be used to test outerplanarity of a graph. The concise outlook of wing-1 graph makes some properties of outerplanar graph can be easier found. For example, every wing-1 graph, i.e. every outerplanar graph, has a degree ≤ 2 vertex. Wing-1 graph pictures a better geometric view

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to make observation for outerplanar graph, which motivates our work on new results of coloring of outerplanar graph.

Our paper is organized as follows. In section 3 we prepare some definitions and introduce wing graph. Section 4 proves the equivalence relation between outerplanar graph and wing-1 graph and deduces a simple linear time algorithm to draw an outerplanar graph to a wing-1 graph. Inspired by the simple structure of wing-1 graph, some properties of outerplanar graph can be easier proved.

Section 5 presents more properties of color assignment of outerplanar graph and discusses a possible way to generalize our results of 3-coloring of outerplanar graph to prove four color problem in planar graph. Section 6 concludes.

3 Wing Graph

In this part, we introduce wing graphs to prepare for the following sections. In a 2-dimension surface, if we draw a straight line, then the surface is split into two parts separately at the left and right side of the line. We define wing graph as follows.

Definition 1. *If a graph is drawn as all its vertices on a straight line without edge crossing on a 2-dimension surface then it is a wing graph.*

Furthermore, we define wing-1 and wing-2 graph as below:

Definition 2. *If a wing graph is drawn as its edges at only one side, then it is a wing-1 graph; If a wing graph can be drawn as its edges at both sides, then it is a wing-2 graph.*

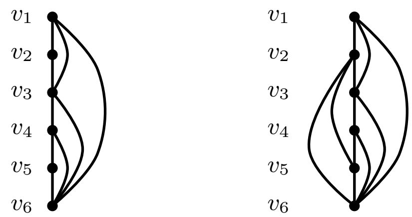
Corresponding to a drawing of a wing graph, all vertices are on a straight line. The two vertices at the ends of the line are called end-vertices; in wing-1 graph, vertices expose to the side of the line where edges are drawn are called outer-vertices; similarly in wing-2 graph, vertices expose to any side of the line are called outer-vertices; Other vertices are called inner-vertices. Trivially, end-vertices are always out-vertices.

Figure 1.a displays a wing-1 graph. All vertices are on the straight vertical line, edges are at right side of the line. $\{v_1, v_6\}$ are end-vertices, also the only two outer-vertices. Figure 1.b displays a wing-2 graph. Edges are at both sides of the straight line. $\{v_1, v_6\}$ are end-vertices, $\{v_1, v_2, v_6\}$ outer-vertices, and $\{v_3, v_4, v_5\}$ inner-vertices.

To further categorize wing graphs, we consider one side of the straight line as a layer or a plane and the line is the boundary of the layer or plane. If at least x layers are needed to draw a graph as no edge crossing, then the graph is a wing-x graph.

Given a graph $G(V, E)$, a set of vertices $U \subseteq V$ and $|U| = x > 0$, if there is a K_x minor in which any vertex $v \in K_x$ is contracted from $S \subseteq V$ and $S \cap U \neq \emptyset$, then we say in G the K_x minor is on U. If $U' \subseteq V$ and $U \subseteq U'$, we also say there is a K_x minor on U'.

In a $K_{x,y}$ minor, we call the x vertices as upper-vertices, and the y vertices lower-vertices. A certain set of vertices is defined as a perimeter trace as below.



(a) (b)

Fig. 1. (a) wing-1 Graph, (b) wing-2 Graph

Definition 3. *Given an outerplanar graph $G(V, E)$, its perimeter trace is a set of vertices $U \subseteq V$ if:*

1. *on U there is no K_3 minor;*
2. *no $K_{2,2}$ minor in which all upper-vertices or all lower-vertices belong to U.*

It is trivial that all subsets of a perimeter trace U of an outerplanar graph G are perimeter traces; if we add a vertex v into G and v is only connected with vertices in U, then the new graph is still outerplanar.

4 Wing-1 Graph

In subsection 4.1, we prove that a graph is outerplanar if and only if it is a wing-1 graph. In subsection 4.2, we show some outerplanar graph's properties are easier proved by wing-1 graph's simple structure. We also find there exists a 3-color assignment for an outerplanar graph satisfies certain constraints.

4.1 Wing-1 and Outerplanar Graph

In this part, we prove Theorem 1 to show an outerplanar graph is a wing-1 graph. Note that a cut set containing more than one vertex implies none of these vertices is a cut vertex. At first we give some preliminary results.

Lemma 1. *In an outerplanar graph, if there is a $K_{2,2}$ minor on 4 vertices, then its upper-vertices or lower-vertices form a cut set.*

Proof. Assume there is a $K_{2,2}$ minor on $U = \{u_1, u_2, u_3, u_4\}$, and no harm to say $\{u_1, u_2\}$ are upper-vertices and $\{u_3, u_4\}$ lower-vertices. If $\{u_1, u_2\}$ is not a cut set, then there is a path $P_{3,4}$ between u_3 and u_4 without passing through $\{u_1, u_2\}$. So $P_{3,4}$ has two cases:

1. $P_{3,4} = e(u_3, u_4)$: if $\{u_3, u_4\}$ is not a cut set, then there is a path $P_{1,2}$ between u_1 and u_2 without passing through $\{u_3, u_4\}$. So on U there is a K_4 minor which contradicts with criterion of outerplanar graph.

2. $P_{3,4} \neq e(u_3, u_4)$: assume there is $u \notin U$ on path $P_{3,4}$, then on $U \cup \{u\}$ there is a $K_{2,3}$ minor which contradicts with criterion of outerplanar graph.

Given a connected outerplanar graph $G(V, E)$, we can always find two vertices u_1, u_2 between which there is a path $P_{1,2} = e(u_1, u_2)$ and $\{u_1, u_2\}$ is not a cut set or $P_{1,2} \neq e(u_1, u_2)$ and every vertex in $P_{1,2} \setminus \{u_1, u_2\}$ is a cut vertex separating u_1 and u_2. For example, two neighbor vertices on a connected outerplanar graph's outer face can be chosen as $\{u_1, u_2\}$. We name vertices in $P_{1,2} \setminus \{u_1, u_2\}$ as bridge vertices of $\{u_1, u_2\}$. Below we use $U = \{u_1, w_1, ..., w_x, u_2\}$ to represent the vertices orderly appearing on a path in an outerplanar graph.

Proposition 1. *In a connected outerplanar graph $G(V, E)$, a set of vertices $U = \{u_1, w_1, ..., w_x, u_2\} \subseteq V$ is a perimeter trace if and only if there are $W \subseteq V$ as $\{u_1, u_2\}$'s bridge vertices and $\{w_1, ..., w_x\} \subseteq W$.*

Proof. If there is no $W \subseteq V$ as $\{u_1, u_2\}$'s bridge vertices, then there is a cut set C with $|C| \geq 2$ separating u_1 and u_2. Then on $C \cup \{u_1, u_2\}$ there is a $K_{2,2}$ minor whose upper-vertices(lower-vertices) are $\{u_1, u_2\}$, so U is not a perimeter trace. If W exists, and there is $w \in \{w_1, ..., w_x\} \setminus W$, then w is not a cut-vertex separating u_1 and u_2, then on U there is a K_3 minor, which means U is not a perimeter trace.

If W are $\{u_1, u_2\}$'s bridge vertices, there is no K_3 minor on $W \cup \{u_1, u_2\}$; if no two vertices in $W \cup \{u_1, u_2\}$ form a cut set, then by Lemma 1 there is no $K_{2,2}$ minor whose upper-vertices(lower-vertices) are only contained in $W \cup \{u_1, u_2\}$. Hence, $W \cup \{u_1, u_2\}$ is a perimeter trace. So $U \subseteq W \cup \{u_1, u_2\}$ is a perimeter trace.

Corollary 1. *In a wing-1 graph, its outer-vertices form a perimeter trace.*

Proof. Since the union of perimeter traces for each component of a disconnected outerplanar graph is a perimeter trace of the whole graph, then it concludes by applying Proposition 1 on the simple structure of wing-1 graph.

Also by Proposition 1, if an outerplanar graph is 2-connected, a perimeter trace can be formed by two arbitrary neighbor vertices on the hamiltonian cycle. In this paper, a n-connected graph means that the graph remains connected after removing arbitrary $(n - 1)$ vertices.

Lemma 2. *Given an outerplanar graph $G(V, E)$, if $U \subseteq V$ is a perimeter trace, and $u \in U$, then $U' = \{\{U \setminus u\} \cup N(u)\}$ in graph $G' = G \setminus u$ is a perimeter trace.*

Proof. There are two cases depending on G is connected or disconnected. When G is disconnected, note that the union of perimeter traces of components of G is G's perimeter trace, which means we can do the proof on the component containing u. Hence below we only prove the case when G is connected. Assume $U = \{u_1, u_2, ..., u_x\}$ orderly appearing on a path in G.

When G is connected, there are two cases: G' is connected or disconnected. By Proposition 1, if $G' = G \setminus u$ is connected, we can assume $u = u_1$ and get two subcases:

1. $deg(u) = 1$, trivially U' is a perimeter trace of G'.
2. $deg(u) > 1$, assume $N(u) = \{q_1, q_2, ..., q_y\}$ orderly appearing on a path from q_1 to u_2. Suppose there is no path $P_{1,2}$ between q_1 and u_2 where $P_{1,2} \setminus \{q_1, u_2\}$ are bridge vertices of $\{q_1, u_2\}$, then there is a cut set C with $|C| \geq 2$ separating q_1 and u_2. Because U is a perimeter trace, by Proposition 1, there is a path P between u_1, u_2, and $P' = P \setminus \{u_1, u_2\}$ are bridge vertices of $\{u_1, u_2\}$. Obviously $C \cap P' = \emptyset$. So on $C \cup \{u_1, q_1, u_2\}$ there is a $K_{2,3}$ minor which is a contradiction. Hence path $P_{1,2}$ exists. Note every vertex in $N(u) \setminus \{U \cup \{q_1\}\}$ is a cut vertex, otherwise there is a K_4 minor on $u \cup N(u)$. So $N(u) \setminus \{U \cup \{q_1\}\} \subseteq P_{1,2}$. By Proposition 1, U' is a perimeter trace of G'.

If $G' = G \setminus u$ is disconnected, then u is a cut vertex of G. Suppose $V = \{v_1, v_2, ...u, ..., v_n\}$ orderly appear on the straight line. This case can be proved similarly by doing induction on two subgraphs of G: $G_1 = G \cap \{v_1, ..., u\}$ and $G_2 = G \cap \{u, ..., v_n\}$. Only need to notice that after deleting u from G_1, G_2 respectively, the union of two perimeter traces of G_1, G_2 is a perimeter trace of G'.

Theorem 1. *Given an outerplanar graph $G(V, E)$ and $U = \{u_1, u_2, ..., u_x\}$ is a perimeter trace, G can be transfer to a wing-1 graph in a way such that u_1 and u_2 are end-vertices, U are outer-vertices, and the order of U on the straight line is kept the same as they are on the outer face of G.*

Proof. We make induction on $|V|$. If $|V| \leq 2$, the conclusion holds trivially. When $|V| = n$, set $G' = G \setminus u_1$. By Lemma 2, $U' = \{U \setminus u_1\} \cup N(u_1)$ is a perimeter trace of G'. Assume $U' = \{q_1, ..., q_y, u_2, ..., u_x\}$, then $N(u_1) \subseteq \{q_1, ..., q_y, u_2\}$.

By induction, G' can be transferred to a wing-1 graph G'_w in which U' are outer-vertices, $\{u'_1, u_2\}$ are the two end-vertices, and the order of U' is kept as that on outer face of G'. By definition 1, we can get a new wing-1 graph G_w by simply adding u_1 before u'_1 into G'_w to be the new end-vertex. Because $N(u_1) \subseteq \{q_1, ..., q_y, u_2\}$ and $\{q_1, ..., q_y, u_2\}$ are outer-vertices of G'_w, adding edges to G_w between u_1 and $N(u_1)$ will produce no edge crossing. Therefore G_w is still a wing-1 graph and now $G_w = G$. Moreover, U are exposed to the outside in G_w which means U are outer vertices of G_w and have the same order as in G's outer face.

Corollary 2. *A graph is outerplanar if and only if it is a wing-1 graph.*

Proof. By Theorem 1, an outerplanar graph is a wing-1 graph, and since that a wing-1 graph is outerplanar trivially holds, so a graph is outplanar if and only if it is a wing-1 graph.

By Proposition 1, we can locate a perimeter trace of an outerplanar graph easily. Theorem 1 can be written as an algorithm to draw a outerplanar graph into a corresponding wing-1 graph and also an outerplanarity-test algorithm. The complexity of the algorithm is $O(n)$ where $n = |V|$ in $G(V, E)$.

4.2 Properties of Wing-1 Graph

In this part, some interesting properties can be easily proved inspired by the simple structure of wing-1 graph. W.L.O.G, assume $V = \{v_1, v_2, ..., v_n\}$ are orderly on the straight line of a wing-1 graph.

Lemma 3. *A Wing-1 graph G has minimum degree ≤ 2.*

Proof. In G if $deg(v_i) \geq 3$, then assume there is edge $e(v_i, v_j)$ where $j > i + 1$. Next we show that among vertices $\{v_{i+1}, ..., v_{j-1}\}$ there is at least a vertex with degree ≤ 2 by doing induction on $(j - i)$. Because $j > (i + 1)$, the base condition is $j - i = 2$ where there are vertices $\{v_i, v_{i+1}, v_j\}$. v_{i+1} can only have edges with v_i, v_j, hence $deg(v_{i+1}) \leq 2$.

If $j - i > 2$, there are three cases:

1. there is edge $e(v_i, v_p)$ where $(i + 1) < p < j$. Hence by induction, among vertices $v_{i+1}, ..., v_{p-1}$ there is a vertex with degree ≤ 2.
2. there is edge $e(v_p, v_j)$ where $i < p < (j - 1)$, this can be proved similarly as above.
3. $deg(v_{i+1}) \leq 2$, otherwise there is edge $e(v_{i+1}, v_p)$ with $(i + 2) < p < j$, by induction, there is one vertex with degree ≤ 2 among vertices $v_{i+2}, ..., v_{p-1}$.

So we prove that G has its minimum degree ≤ 2.

In the proof of Lemma 3, we infer there is at least a degree ≤ 2 vertex between every two outer-vertices. In fact it is not difficult to generalize as there are at least $(x - 1)$ degree ≤ 2 vertices if there exist x outer-vertices.

For convenience, we give a definition as below:

Definition 4. *Given a wing-1 graph $G(V, E)$, $W \subseteq V$ is called a vertex-related set, if for any $\{v_i, v_j\} \in W (i < j)$, there is no edge between a vertex between $\{v_i, v_j\}$ and a vertex in $V \backslash \{v_i, v_j\}$.*

For example, in Figure 1.*a* $\{v_1, v_2, v_3\}$ and $\{v_3, v_4, v_6\}$ are two vertex-related sets. W.L.O.G, assume every vertex-related set is maximal.

Observation 11 *In a wing-1 graph $G(V, E)$, if a vertex $v \in V$ has $deg(v) \leq 2$, then $v \cup N(v)$ can only belong to one vertex-related set R. Moreover, when $deg(v) = 2$ $N(v)$ belongs to a different vertex-related set from R if and only if there is no edge between v's two neighbor vertices.*

Theorem 2. *A connected wing-1 graph $G(V, E)$ has a 3-color assignment cl. Furthermore, if in cl there is a vertex-related set R forming a K_x division, then an independent set of vertices $I \subseteq R$ can be colored with one color, and*

1. *if $x = 3$: $\{R \setminus I\}$ are colored with the other two colors and the picked independent set can not be empty unless $|R|$ is even.*
2. *if $x = 2$: $\{R \setminus I\}$ are colored with the other two colors and the picked independent set can be empty.*

Proof. This theorem can be proved by doing induction on $|V|$. By Lemma 3, in G there is a degree ≤ 2 vertex $v \in V$. If we delete the vertex v, we get a new graph with less vertices, on which we can apply induction. Observation 11 can help to simplify the case study. This proof is a little long but not hard to follow. We put the details in Appendix.

5 Generalizable Coloring of Wing-1 Graph

In this Section, we prove a wing-1 graph, i.e. an outerplanar graph, has a 3-color assignment satisfying certain coloring constraints. More important, we find it promising to generalize the results to coloring a planar graph. At first we define *series* and *cluster* introduced in [8]. A sequence of vertices appearing along a simple path in one direction form a *seires*. After deleting vertices in a *series* of a path, the left *series* is still called a *series* of the path. A continuous part of a *series* is called a *cluster*.

If we decompose a *series* S into a sequence of clusters $\Upsilon = \{\Upsilon_1, \Upsilon_2, ..., \Upsilon_x\}$ by order and $\bigcup_{i=1}^{x} \Upsilon_i = S$, Υ is called clusters of the series S, in which the end of $\Upsilon_i (1 \leq i < x)$ only overlap on ≤ 1 vertex with the beginning of Υ_{i+1}. So there may be more than one *cluster* on one vertex. Definitions of *series* and *cluster* can be generalized on a set of vertices $U = \{u_1, u_2, ..., u_x\}$ if u_i, u_{i+1} are connected and there is a path P_i between u_i, u_{i+1} satisfying $P_i \cap U = \{u_i, u_{i+1}\}$ for $1 \leq i < x$. If $u_i, u_{i+1} (1 \leq i < x)$ are disconnected we assume an edge $e(u_i, u_{i+1})$ which is certainly a path P_i.

Assume vertices of a wing-1 graph G orderly appear on the straight line as $V = \{v_1, v_2, ..., v_n\}$ and clusters on ordered set of G's outer-vertices are $\Upsilon = \{\Upsilon_1, \Upsilon_2, ..., \Upsilon_x\}$. Name Υ_i and $\Upsilon_{i+1} (1 \leq i < q)$ two neighbor clusters.

To a *cluster* Υ_i, we add a vertex γ_i with $N(\gamma_i) = \Upsilon_i$, and name γ as Υ_i's cluster-vertex. In Figure 2.a, $\{\Upsilon_1, ..., \Upsilon_6\}$ are clusters defined on vertex $\{v\}$ whose cluster-vertices are corresponding to $\{\gamma_1, ..., \gamma_6\}$. In Figure 2.$b$, $\{v_1, v_3, v_6\}$ is a series of outer-vertices which is decomposed into clusters $\{\Upsilon_1, ..., \Upsilon_8\}$. Specifically, $\Upsilon_1 = \Upsilon_2 = \{v_1\}$, $\Upsilon_3 = \{v_1, v_3\}$, $\Upsilon_4 = \{v_3\}$, $\Upsilon_5 = \{v_3, v_6\}$, $\Upsilon_6 = \Upsilon_7 = \Upsilon_8 = \{v_6\}$.

Below assume colors $CL = \{1, 2, 3\}$ are used, and a color collection is $\{1, 2\}$,$\{1, 3\}$, or $\{2, 3\}$. Color $\{1\}$ is equivalent with collection $\{1, 2\}$, $\{1, 3\}$ or collections $\{\{1, 2\}, \{1, 3\}\}$, so is color $\{2\}$ and color $\{3\}$. When only 3 colors are used, a *cluster* Υ_i is colored with the collection cn, if and only if its cluster-vertex γ_i is colored with $CL \setminus cn$. For example, $color(\Upsilon_i) = \{1, 2\}$ is equivalent with $color(\gamma_i) = \{3\}$. Below $cn(\Upsilon_i)$ represents the collection of colors used on Υ_i.

Given two sets of collection cn_1, cn_2, if $cn_1 \setminus cn_2 \neq \emptyset$ and $cn_2 \setminus cn_1 \neq \emptyset$, we say cn_1, cn_2 are inconsistent; otherwise, they are consistent. For example, $\{1, 2\}$ and $\{\{1, 2\}, \{1, 3\}\}$ are consistent, but inconsistent with $\{\{1, 3\}, \{2, 3\}\}$.

Definition 5. *In a wing-1 graph $G(V, E)$, U are outer vertices, rules of collection-constraints of coloring on U are defined following:*

1. *a cluster is colored with colors from a collection;*
2. *two neighbor clusters have the same or different collections.*

Definition 6. *In a wing-1 graph G, collection-constraints ct are defined on its outer-vertices, define constraint-graph G_{ct} by extending G as following: 1)every cluster has a cluster-vertex, and if $cn(\Upsilon_i) = cn(\Upsilon_j)$ the two clusters share a cluster-vertex; 2)if $cn(\Upsilon_i) \neq cn(\Upsilon_j)$, there is edge $e(\gamma_i, \gamma_j)$.*

Cluster-vertices(γ-vertices) can have an order according to that of their belonging clusters. In Figure 2.a, the constraint-graph by constraints $cn(\Upsilon_1) \neq cn(\Upsilon_2)$ and $cn(\Upsilon_2) \neq cn(\Upsilon_3)$ is displayed. If a more constraint $cn(\Upsilon_5) = cn(\Upsilon_6)$ is added, vertices $\{\gamma_5, \gamma_6\}$ will be merged into a new vertex $\gamma_{5,6}$. Figure 2.b is another similar example of constraint-graph, and its corresponding constraints can be found easily.

Observation 12 *$G_{ct} \setminus G$ is a K_2 or K_1 division, and G_{ct} is a wing-1 graph.*

Proof. Obviously $G_{ct} \setminus G$ can only be a K_2 or K_1 division. Turn around the K_2 or K_1 division to an end of G in G_{ct} then the resulted graph is a wing-1 graph.

γ-vertices can be divided into continuous parts each of which is a K_2 or K_1 division. Similar as neighbor clusters, we can define neighbor divisions formed by γ-vertices. Two neighbor divisions can overlap on at most one *cluster*. Since a *cluster* is corresponding to a γ-vertex, without causing confusion below sometimes we refer a division as the clusters that corresponding to γ-vertices that form the division.

Definition 7. *Rules of division-constraints are defined following:*

1. *a division has a set of ≤ 2 collections.*
2. *two neighbor divisions have inconsistent sets of collections.*
3. *all clusters on a vertex belong to the same division.*
4. *if there is an edge $e(v_1, v_2)$ and the two divisions of clusters on v_1, v_2 are both K_2 division, they are neighbor divisions.*

Observation 13 *There is a color assignment of G_{ct} uses ≤ 3 colors, if and only if there is a color assignment cl of G using ≤ 3 colors and satisfying collection-constraints, and $3, 4$ cases of division-constraints.*

Proof. cl_{ct} can be used on G to be cl. cl satisfies all collection-constraints following definitions of collection-constraints and constraint-graph. If vertex v is colored with $\{1\}$, then cluster-vertices connected with $\{v\}$ are colored with $\{2, 3\}$. Hence they can belong to the same division. If there is edge $e(v_1, v_2)$, and $\{v_1, v_2\}$ colored with $\{1, 2\}$ respectively, then two K_2 divisions of v_1, v_2 can be colored with $\{2, 3\}$ and $\{1, 3\}$ respectively, whose corresponding collections are inconsistent. So $3, 4$ cases of division-constraints are satisfied. The reverse can be proved similarly.

Since G_{ct} is a wing-1 graph, by Theorem 2 there always exists such a cl_{ct}. In Figure 2.b, if there is edge $e(v_3, v_6)$, $\{\gamma_3, \gamma_4, \gamma_5\}$ and $\{\gamma_5, \gamma_6, \gamma_7, \gamma_8\}$ can not belong to the same division. If there is no edge $e(v_3, v_6)$, $\{\gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\}$ can belong to the same division.

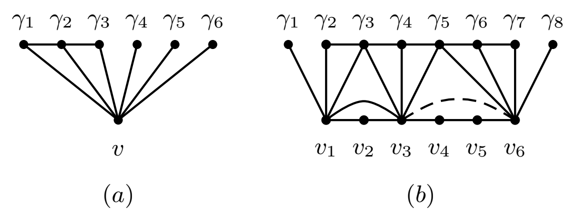


Fig. 2. Wing-1 Graph and Constraint-graph

Theorem 3. *Given a wing-1 graph $G(V,E)$ and its outer-vertices U, G has a color assignment cl using ≤ 3 colors when it satisfies all collection-constraints and division-constraints.*

Proof. We make induction on $|V|$. When $|V| = 1$, suppose $V = \{v\}$. The constraint-graph G_{ct} defined by collection-constraints ct have $\{v\}$ and all γ vertices colored with $\{1\}$ and $\{2,3\}$ respectively. Hence division-constraints can be verified as satisfied. Then by Observation 13, the conclusion holds when $|V| = 1$.

When $|V| = n$, after removing a vertex, say v_n, the graph $G' = G \setminus v_n$ is a wing-1 graph and $U' = \{U \setminus v_n\} \cup \{N(v_n)\}$ are outer-vertices. In G' clusters only on $\{v_n\}$ disappear and a new *cluster* $\Upsilon_{N(v_n)}$ on $N(v_n)$ appears whose cluster-vertex $\gamma_{N(u)} = v_n$. Define collection-constraints and division-constraints on U' inheriting from U. And if there is *cluster* Υ_i with $\{v_n\} \subset \Upsilon_i$, then set $cn(\Upsilon_{N(v_n)}) \neq cn(\Upsilon_i)$. By doing this, the constraint-graph of G' is a induced subgraph of G_{ct}.

By induction, G' has a ≤ 3 color assignment cl' satisfying all constraints on U'. cl' can be extended to be a color assignment cl for G by setting $color_{cl}(v_n) = CL \setminus cn(\Upsilon_{N(v_n)})$. Assume $cn(\Upsilon_{N(u)}) = \{1,2\}$, then $color_{cl}(v_n) = \{3\}$. So collections of clusters on $\{v_n\}$ can be $\{1,3\}$ or $\{2,3\}$. Hence γ vertices of clusters on $\{v_n\}$ can be colored with $2,1$ respectively. Because of the relation between constraint-graphs of G' and G, cl can be extended to be a ≤ 3 color assignment of G_{ct}. By Observation 13, next we need to show division-constraints can be satisfied by cl.

Let D_{v_n} be the division of clusters on $\{v_n\}$ in G. D_{v_n} can have collections from $\{\{1,3\},\{2,3\}\}$. Name division D' of clusters formed by tail part of outer-vertices in G'. If in G D_{v_n}, D' belong to the same(different) division, then in G', make $\Upsilon_{N(u)}, D'$ belong to different(the same) divisions. These actions are irrelevant to above proof. If division D_{v_n}, D' in G belong to:

1. different divisions: then in G', $\Upsilon_{N(u)}$ and D' belong to the same division. Because $\Upsilon_{N(u)}$ has collection $\{\{1,2\}\}$, D' can have collections $\{\{1,2\}\}$, $\{\{1,2\},\{1,3\}\}$ or $\{\{1,2\},\{2,3\}\}$, all of which are inconsistent with $\{\{1,3\},\{2,3\}\}$.
2. the same division: then in G', $\Upsilon_{N(u)}$ and D' belong to different divisions. I.e., possible collections of D' are from $\{\{1,3\},\{2,3\}\}$, which are the same as possible collections of D_{v_n}.

The other division-constraints are satisfied because of cl', so we can conclude.

Corollary 3. *A wing-1 graph has a color assignment using* ≤ 3 *colors and outer-vertices are assigned* ≤ 2 *colors.*

Proof. We treat all the outer-vertices as a *cluster*. Then by Theorem 3, there is a color assignment using ≤ 3 colors and this *cluster* is colored with a collection of 2 colors.

The definition of a perimeter trace in an outerplanar graph and Theorem 2 can lead to the same result shown in Theorem 3. However, we choose to give a proof in a way that is longer but easier to generalize.

There are close inner relations among outerplanar graphs, planar graphs, and Hadwiger conjecture. Hadwiger conjecture when $k = 4$ case states that if a graph has no K_4 minor, its chromatic number is 3 [5], so an outerplanar graph which has no K_4 minor can be colored by 3 colors.

The $k = 5$ case of hadwiger conjecture states that if a graph has no K_5 minor, it is 4-colorable. An alternative way to prove it is to prove a planar graph which has no K_5 minor is 4-colorable. We show many coloring constraints can be satisfied for coloring an outerplanar graph in this paper. In fact, such a methodology can be used on planar graphs which is promising leading to four color theorem or even arbitrary k of hadiwiger conjecture.

Follow-up work has been done. In [8], a perimeter trace in planar graph is defined similarly as for outerplanar graph and is proved to be boundary of an outer face. Also a similar result as Lemma 2 has been proved, hence the induction method used in proof of Theorem 3 can be applied for planar graph. What is more interesting, the extended subgraph in Definition 6is an union of path graphs in outerplanar graph, while it is an outerplanar graph in planar graph. The next step to do generalization is to well define similar but more complex rules of collection-constraints and division-constraints on planar graph.

6 Conclusion

In this paper, we find a kind of graphs named wing graphs. They have clear geometric view and good categorized property. Firstly we prove outerplanar graphs are equivalent with wing-1 graphs. The conclusion gives a tool to draw an outerplanar graph to a wing-1 graph which is also a outerplanarity-test tool. Based on wing-1 graph, we prove that an outerplanar graph has a special 3-coloring assignment satisfying special constraints. In the future, we expect to generalize our result to show similar properties of planar graph. In another direction, since a wing-2 graph is obviously planar, it would be interesting to find out its minor criterion and even further study wing-x graphs.

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7 Appendix

Theorem 2 *A connected wing-1 graph $G(V, E)$ has a 3-color assignment cl, and if in cl there is a vertex-related set R forming a K_x division, then an independent set of vertices $I \subseteq R$ can be colored with one color, and*

1. *if $x = 3$: $\{R \setminus I\}$ are colored with the other two colors and the picked independent set can not be empty unless $|R|$ is even.*
2. *if $x = 2$: $\{R \setminus I\}$ are colored with the other two colors and the picked independent set can be empty.*

Proof. We make induction on $|V|$. Assume we use colors $CL = \{1, 2, 3\}$. If $|V| = 1$, it holds trivially. If $|V| = n$, by Lemma 3, there is a degree ≤ 2 vertex v in V. Here we assume $deg(v) = 2$, as the case $deg(v) = 1$ can be proved similarly and easier. Assume $N(v) = \{v_1, v_2\}$, by Observation 11, $\{v, v_1, v_2\}$ belongs to a vertex-related set R, and it is evident that R can only consist K_2 or K_3 division.

1. R is on a K_2 division, after breaking symmetry, there are three subcases:
   (a) if $v \in I$ or $\{v, v_1, v_2\} \cap I = \emptyset$ or $I = \emptyset$, contract $\{v, v_1, v_2\}$ to be vertex v' to get graph G'. Because of wing-1 graph's structure, G' is a wing-1 graph. By Observation 11, only the vertex-related set R is changed to be $\{R \setminus \{v, v_1, v_2\}\} \cup \{v'\}$, and I is changed to be $I \setminus \{v\}$. By induction, the new graph has a color assignment cl' satisfying the conclusion. To extend cl' to be a color assignment cl of G, set $color_{cl}(v_1) = color_{cl}(v_2) = color_{cl'}(v')$. Because $color_{cl}(v)$ can be arbitrary color except $color_{cl}(v')$, we can set $color_{cl}(v) = color_{cl'}(I)$ when $v \in I$; and $color_{cl}(v) \neq color_{cl'}(I)$ when $v \notin I$.
   (b) if $\{v_1, v_2\} \subseteq I$, do the same thing as above, except I is changed to be $\{I \setminus \{v\}\} \cup \{v'\}$.

(c) if $v_1 \in I$, delete v and add edge $e(v_1, v_2)$ to get a new wing-1 graph G'. Then only the vertex-related set R is changed to be $\{R \setminus \{v\}\}$, and I is unchanged. By induction, the new graph has a color assignment cl' satisfying the conclusion. To extend cl' to be required color assignment cl of G, set $color_{cl}(v) = CL \setminus color_{cl'}(\{v_1, v_2\})$.

It is not difficult to verify that the color assignment cl of G satisfy the conclusion.

2. R is on a K_3 division. By Observation 11, if there is edge $e(v_1, v_2)$, the conclusion holds trivially. So we assume there is no edge $e(v_1, v_2)$, which means that $\{v_1, v_2\}$ can only belong to R simultaneously. Then this case can be proved similarly as K_2 division case, and only need to notice that when $|R|$ is odd, $I \neq \emptyset$.